NATURAL FREQUENCIES OF AN INFINITE CYLINDER WITH PLANE STRAIN

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November 3, 2004

Variables

u	is the axial displacement
v	is the tangential displacement
w	is the radial displacement
E	is the modulus of elasticity
R	is the radius
ρ	is the mass/volume
ν	is the Poisson ratio
c	is the speed of sound in the material
t	is time
h	is the wall thickness
ω	is the natural frequency
n	is an index, n=0,1,2,…
k	is a nondimensional thickness factor
Ω	is a nondimensional natural frequency
B	Eigenvector scale factor
C	Eigenvector scale factor

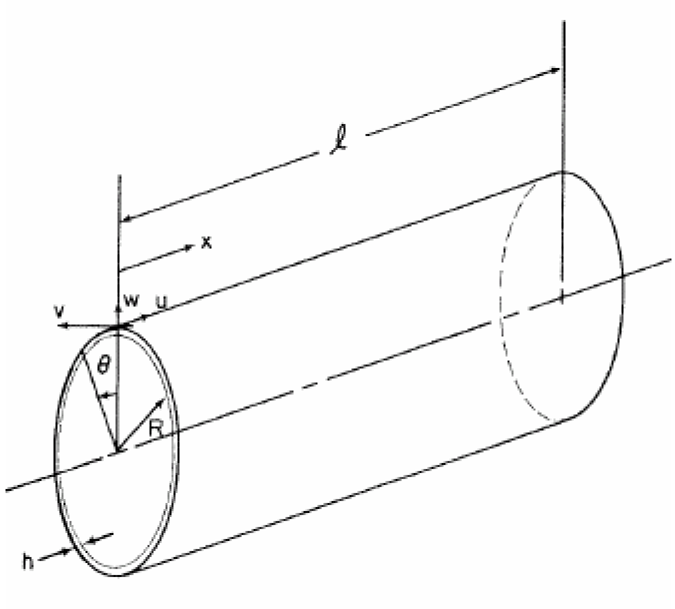

Figure 1. Cylinder Diagram

The three translation variables are

$$u = 0 \tag{1}$$

$$v = v(\theta) \tag{2}$$

$$w = w(\theta) \tag{3}$$

There are two coupled equations of motion, taken from Reference 1.

$$\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial w}{\partial \theta} = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 v}{\partial t^2} \tag{4}$$

$$\frac{\partial v}{\partial \theta} + \left[1 + k\left(1 + \frac{\partial^2}{\partial \theta^2}\right)^2\right] w = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{5}$$

where

$$k = \frac{h^2}{12R^2} \tag{6}$$

Equations (4) and (5) together model both the bending and membrane response.

$$\frac{\partial v}{\partial \theta} + \left[1 + k\left(\frac{\partial^4}{\partial \theta^4} + 2\frac{\partial^2}{\partial \theta^2} + 1 \right) \right] w = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{7}$$

$$\frac{\partial v}{\partial \theta} + \left[k\left(\frac{\partial^4 w}{\partial \theta^4} + 2\frac{\partial^2 w}{\partial \theta^2} + w \right) + w \right] = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{8}$$

$$\frac{\partial v}{\partial \theta} + \left[k\frac{\partial^4 w}{\partial \theta^4} + 2k\frac{\partial^2 w}{\partial \theta^2} + w(k+1) \right] = \frac{\rho\left(1-\nu^2\right)R^2}{E}\frac{\partial^2 w}{\partial t^2} \tag{9}$$

Assume that the tangential displacement is

$$v = B\sin(n\theta)\cos(\omega t) \tag{10}$$

$$\frac{\partial v}{\partial \theta} = nB\cos(n\theta)\cos(\omega t) \tag{11}$$

$$\frac{\partial^2 v}{\partial \theta^2} = -n^2 B\sin(n\theta)\cos(\omega t) \tag{12}$$

$$\frac{\partial v}{\partial t} = -\omega B \sin(n\theta)\sin(\omega t) \tag{13}$$

$$\frac{\partial^2 v}{\partial t^2} = -\omega^2 B \sin(n\theta)\cos(\omega t) \tag{14}$$

Assume that the radial displacement is

$$w = C\cos(n\theta)\cos(\omega t) \tag{15}$$

$$\frac{\partial}{\partial \theta} w = -n\,C\sin(n\theta)\cos(\omega t) \tag{16}$$

$$\frac{\partial^2}{\partial \theta^2} w = -n^2 C\cos(n\theta)\cos(\omega t) \tag{17}$$

$$\frac{\partial^3}{\partial \theta^3} w = n^3 C\sin(n\theta)\cos(\omega t) \tag{18}$$

$$\frac{\partial^4}{\partial \theta^4} w = n^4 C\cos(n\theta)\cos(\omega t) \tag{19}$$

$$\frac{\partial}{\partial t} w = -\omega C\cos(n\theta)\sin(\omega t) \tag{20}$$

$$\frac{\partial^2}{\partial t^2} w = -\omega^2 C\cos(n\theta)\cos(\omega t) \tag{21}$$

Substitute the assumed solutions into equation (4).

$$-n^2 B \sin(n\theta)\cos(\omega t) \;\; - nC\sin(n\theta)\cos(\omega t) \;\; = -\frac{\rho\left(1-\nu^2\right)R^2}{E}\omega^2 B\sin(n\theta)\cos(\omega t) \tag{22}$$

$$-n^2 B \; - nC \;\; = -\frac{\rho\left(1-\nu^2\right)R^2}{E}\omega^2 B \tag{23}$$

$$\left[\frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E} - n^2\right] B \; - nC \;\; = 0 \tag{24}$$

Let

$$\Omega^2 \;\; = \;\; \left[\frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}\right] \tag{25}$$

$$\left[\Omega^2 - n^2\right] B \; - nC \;\; = 0 \tag{26}$$

$$\left[n^2 - \Omega^2\right] B \; + nC \;\; = 0 \tag{27}$$

Substitute the assumed solutions into equation (5).

$$n\,B\cos(n\theta)\cos(\omega t) + n^4 k\,C\cos(n\theta)\cos(\omega t)$$

$$-2n^2 k\,C\cos(n\theta)\cos(\omega t) + (k+1)C\cos(n\theta)\cos(\omega t)$$

$$= \frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}C\cos(n\theta)\cos(\omega t)$$

(28)

$$n\,B + n^4 k\,C - 2n^2 k\,C + (k+1)C = \frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}C \tag{29}$$

$$n\,B + \left[n^4 k - 2n^2 k + (k+1) - \frac{\omega^2\rho\left(1-\nu^2\right)R^2}{E}\right]C = 0 \tag{30}$$

$$n\,B + \left[n^4 k - 2n^2 k + k + 1 - \Omega^2\right]C = 0 \tag{31}$$

$$\begin{bmatrix} \left(n^2-\Omega^2\right) & n \\ n & n^4 k - 2n^2 k + k + 1 - \Omega^2 \end{bmatrix}\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(32)

$$\begin{bmatrix} \left(n^2 - \Omega^2\right) & n \\ n & 1 + k\left(1 - n^2\right)^2 - \Omega^2 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{33}$$

$$\left(1 + k\left(1 - n^2\right)^2 - \Omega^2\right)\left(n^2 - \Omega^2\right) - n^2 = 0 \tag{34}$$

For n = 0,

$$\left(1 + k - \Omega^2\right)\left(-\Omega^2\right) = 0 \tag{35}$$

$$\Omega^2 = 0 \quad \text{or} \quad \Omega^2 = 1 + k \tag{36}$$

The root $\Omega^2 = 0$ for $n = 0$ corresponds to rigid-body motion of the shell.

For n = 1,

$$\left(1 - \Omega^2\right)\left(1 - \Omega^2\right) - 1 = 0 \tag{37}$$

$$\Omega^4 - 2\Omega^2 + 1 - 1 = 0 \tag{38}$$

$$\Omega^4 - 2\Omega^2 = 0 \tag{39}$$

$$\Omega^2\left(\Omega^2 - 2\right) = 0 \tag{40}$$

$$\Omega^2 = 0 \quad \text{or} \quad \Omega^2 = 2 \tag{41}$$

The root $\Omega^2 = 0$ for $n = 1$ corresponds to rigid-body motion of the shell.

In general,

$$\left(1 + k\left(1 - n^2\right)^2 - \Omega^2\right)\left(n^2 - \Omega^2\right) - n^2 = 0 \tag{42}$$

$$\Omega^4 - \left(1 + n^2 + k\left(1 - n^2\right)^2\right)\Omega^2 + n^2\left(1 + k\left(1 - n^2\right)^2\right) - n^2 = 0 \tag{43}$$

$$\Omega^4 - \left(1 + n^2 + k\left(1 - n^2\right)^2\right)\Omega^2 + k\,n^2\left(1 - n^2\right)^2 = 0 \tag{44}$$

$$\Omega^2 = \frac{1}{2}\left[\left(1 + n^2 + k\left(1 - n^2\right)^2\right) \pm \sqrt{\left(1 + n^2 + k\left(1 - n^2\right)^2\right)^2 - 4k\,n^2\left(1 - n^2\right)^2}\,\right] \tag{45}$$

<u>Reference</u>

1. Leissa, Vibration of Shells, NASA SP-288, Washington, D.C., 1973. (See section 2.2).

APPENDIX A

<u>Example</u>

Consider an infinitely long cylinder with the following properties:

Radius	19 inch
Skin Thickness	0.080 inch
Skin Material	Titanium

The speed of sound in titanium is c = 194,650 in/sec.

The mass density is ρ = 0.16 lbm/in^3.

The elastic modulus is E =1.57e+07 lbf/in^2.

The natural frequencies are calculated using equations (25) and (45). The natural frequencies are shown in the table on the next page. Diagrams of selected mode shapes are given after the table.

There are two frequencies per n value.

n	Lower Frequency (Hz)	Upper Frequency (Hz)
0	0	1727
1	0	2443
2	5.63	3862
3	15.9	5462
4	30.6	7122
5	49.4	8807
6	72.5	10510
7	100	12210
8	131	13930
9	167	15640
10	207	17360
11	251	19080
12	299	20800
13	352	22520
14	408	24240
15	469	25970
16	534	27690
17	604	29410
18	677	31140
19	755	32860
20	837	34590

The dashed line is the undeformed cylinder cross-section in each of the following figures. The solid line is the mode shape.

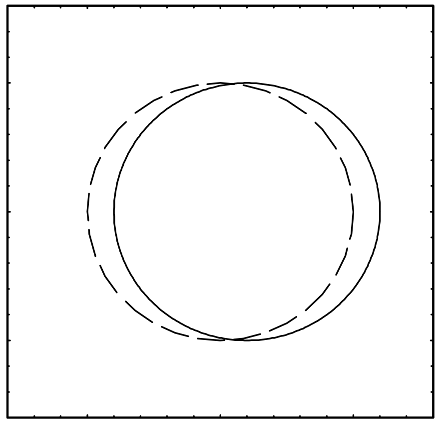

Figure A-1. n=1, Freq = 0 Hz, Rigid-body Motion

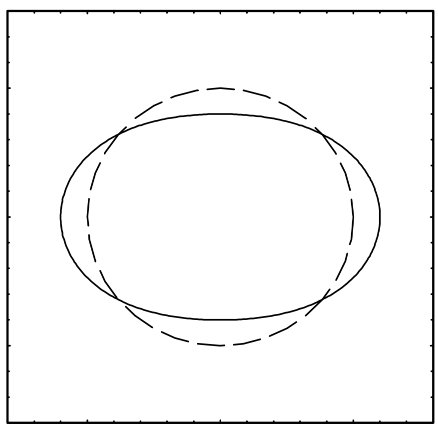

Figure A-2. n=2, Freq = 5.63 Hz, Flexural Mode

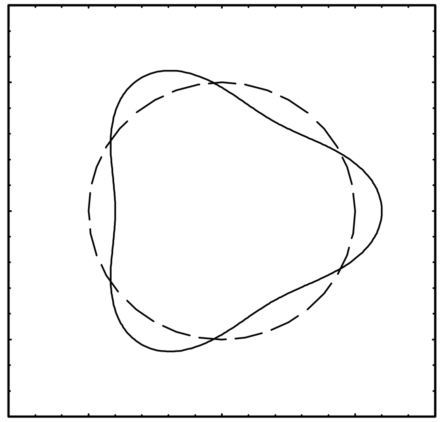

Figure A-3. n=3, Freq = 15.9 Hz, Flexural Mode

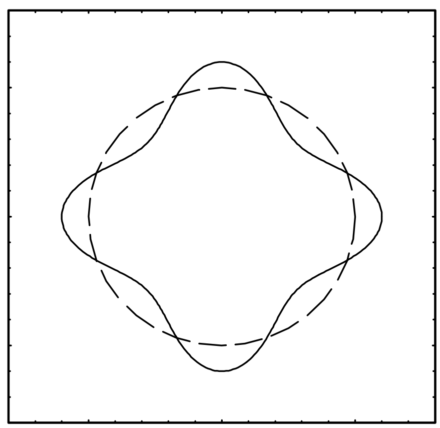

Figure A-4. n=4, Freq = 30.6 Hz, Flexural Mode

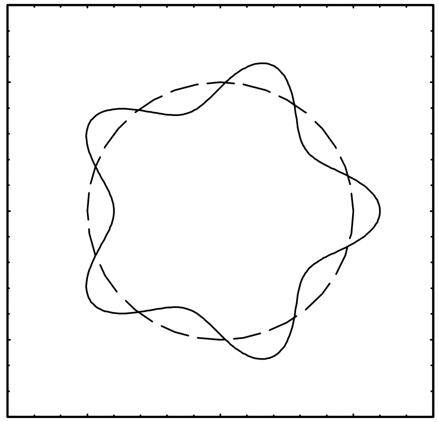

Figure A-5. n=5, Freq = 30.6 Hz, Flexural Mode

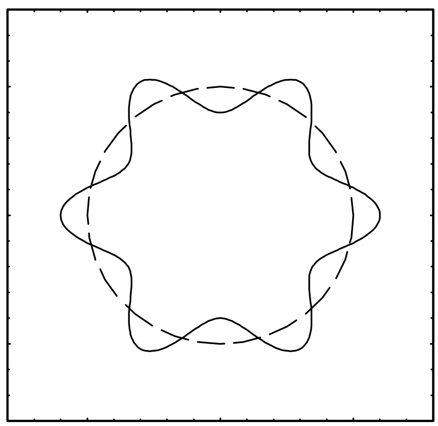

Figure A-6. n=6, Freq = 49.4 Hz, Flexural Mode

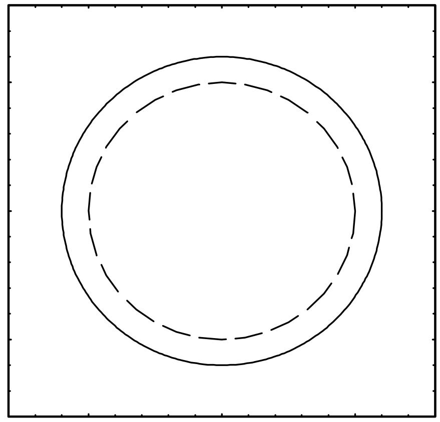

Figure A-7. n=0, Freq = 1727 Hz, Extension Mode

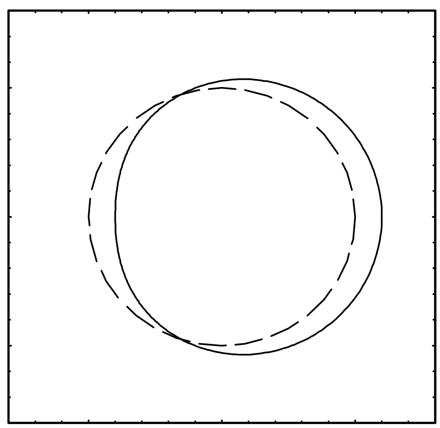

Figure A-8. n=1, Freq = 2443 Hz, Extension Mode