# INITIAL VELOCITY EXCITATION OF THE LONGITUNDINAL MODES IN A BEAM FOR PYROTECHNIC SHOCK SIMULATION
Revision C

By Tom Irvine
Email: tom@irvinemail.org

June 7, 2024

---

## Introduction

Avionics components in launch vehicles must withstand pyrotechnic shock from various sources, particularly stage separation. These components must be subjected to shock testing in the lab to verify design integrity prior to flight.[1]

The shock test method typically uses mechanical excitation or an actual pyrotechnic device such as detonation cord.

## Structural Excitation

Note that there are four common methods for structural excitation:

1. Applied force
2. Base Excitation
3. Flutter or self-excitation
4. Initial velocity or displacement

Applied force test examples include using a hammer or a pneumatic device to excite a beam or plate. The use of pyrotechnic charges is also in this group.

Base excitation examples include shaker shock. Another is a drop tower configured to apply a half-sine pulse of specified amplitude and duration.

Flutter and self-excitation are not applicable to shock testing.

The fourth category of initial velocity or displacement[2] represents a potential method that has apparently not been utilized for previous component testing.

---

[1] The qualification unit itself is not flown, except in the dubious case of proto-qualification units.

[2] As an aside, initial displacement excitation has been used for civil engineering structures. For example, a load cell may be used to apply a static force to a bridge, causing an initial displacement.

Initial Velocity Excitation of Beam's Longitudinal Modes

The following example is taken from Reference 1, pages 372-373.

> A bar moving along the x axis with constant velocity v is suddenly stopped at the end x = 0, so that the initial conditions are $(u)_{t=0} = 0$ and $(\dot{u})_{t=0} = v$. Determine the ensuing vibrations.

The example effectively assumes that beam's boundary conditions are free-free prior to the stopping event. The boundary conditions then immediately become fixed-free at the onset of the stop.

The resulting displacement u(x, t) for a fix-free beam per References 1 and 2 is

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \left\{ \left[ \sin\left(\frac{n\pi x}{2L}\right) \right] \left[ A_n \sin\left(\frac{n\pi ct}{2L}\right) + B_n \cos\left(\frac{n\pi ct}{2L}\right) \right] \right\} \quad (1)$$

where

- L is the length
- c is the longitudinal wave speed
- $\omega_n$ is the natural frequency of mode n

The displacement initial condition yields $B_n = 0$.

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \left\{ \left[ \sin\left(\frac{n\pi x}{2L}\right) \right] \left[ A_n \sin\left(\frac{n\pi ct}{2L}\right) \right] \right\} \quad (2)$$

---

The load is then suddenly released. The resulting free vibration is then measured to determine modal characteristics.

The velocity is thus

$$\dot{u}(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \left\{ \left( \frac{n\pi c}{2L} \right) \left[ \sin\left( \frac{n\pi x}{2L} \right) \right] \left[ A_n \cos\left( \frac{n\pi ct}{2L} \right) \right] \right\} \tag{3}$$

The velocity initial condition is found via the following steps.

$$\sum_{n=1,3,5,\ldots}^{\infty} \left\{ A_n \left( \frac{n\pi c}{2L} \right) \left[ \sin\left( \frac{n\pi x}{2L} \right) \right] \right\} = v \tag{4}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \left\{ A_n \left( \frac{n\pi c}{2L} \right) \left[ \sin\left( \frac{n\pi x}{2L} \right) \right] \left[ \sin\left( \frac{m\pi x}{2L} \right) \right] \right\} = v \left[ \sin\left( \frac{m\pi x}{2L} \right) \right] \quad , \ m=1, 3, 5, \ldots. \tag{5}$$

Integrate

$$\int_0^L \sum_{n=1,3,5,\ldots}^{\infty} \left\{ A_n \left( \frac{n\pi c}{2L} \right) \left[ \sin\left( \frac{n\pi x}{2L} \right) \right] \left[ \sin\left( \frac{m\pi x}{2L} \right) \right] \right\} dx = \int_0^L v \left[ \cos\left( \frac{m\pi x}{2L} \right) \right] dx \tag{6}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \left\{ A_n \left( \frac{n\pi c}{2L} \right) \int_0^L \left[ \sin\left( \frac{n\pi x}{2L} \right) \right] \left[ \sin\left( \frac{m\pi x}{2L} \right) \right] dx \right\} = \int_0^L v \left[ \sin\left( \frac{m\pi x}{2L} \right) \right] dx \tag{7}$$

Orthogonality requires that m=n.

$$A_n \left( \frac{n\pi c}{2L} \right) \int_0^L \left[ \sin^2\left( \frac{n\pi x}{2L} \right) \right] dx = \int_0^L v \left[ \sin\left( \frac{n\pi x}{2L} \right) \right] dx \quad , \ n=1, 3, 5, \ldots. \tag{8}$$

$$A_n \left( \frac{n\pi c}{4L} \right) \int_0^L \left[ 1 - \cos\left( \frac{n\pi x}{L} \right) \right] dx = \left( \frac{-2vL}{n\pi} \right) \cos\left( \frac{n\pi x}{2L} \right) \Bigg|_0^L \tag{9}$$

$$A_n\left(\frac{n\pi c}{4L}\right)\left[x-\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L=\left(\frac{-2vL}{n\pi}\right)\cos\left(\frac{n\pi x}{2L}\right)\Bigg|_0^L \tag{10}$$

$$A_n\left(\frac{n\pi c}{4}\right)=\left(\frac{2vL}{n\pi}\right) \tag{11}$$

$$A_n=\left(\frac{8vL}{n^2\pi^2 c}\right) \tag{12}$$

The displacement is thus

$$u(x,t)=\frac{8vL}{\pi^2 c}\sum_{n=1,3,5,\ldots}^{\infty}\left\{\frac{1}{n^2}\sin\left(\frac{n\pi x}{2L}\right)\sin(\omega_n t)\right\} \tag{13}$$

where

$$\omega_n=n\pi\frac{c}{2L} \tag{14}$$

The corresponding acceleration is

$$\ddot{u}(x,t)=\frac{-8vL}{\pi^2 c}\sum_{n=1,3,5,\ldots}^{\infty}\left\{\frac{(\omega_n)^2}{n^2}\sin\left(\frac{n\pi x}{2L}\right)\sin(\omega_n t)\right\} \tag{15}$$

Now consider only the acceleration at x=L.

$$\ddot{u}(L,t)=\frac{-8vL}{\pi^2 c}\sum_{n=1,3,5,\ldots}^{\infty}\left\{\frac{(\omega_n)^2}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin(\omega_n t)\right\} \tag{16}$$

Damping has been previously neglected. Assume light-damping. With some relaxation of mathematical rigor, the acceleration at x = L becomes

$$\ddot{u}(L,t) = \frac{-8\,v\,L}{\pi^2\,c} \sum_{n=1,3,5,\ldots}^{\infty} \left\{ \frac{(\omega_n)^2}{n^2} \exp(-\xi_n\,\omega_n\,t) \sin\left(\frac{n\,\pi}{2}\right) \sin(\omega_n\,t) \right\} \tag{17}$$

where $\xi$ is the damping ratio.

<u>Example</u>

A steel beam (or rod, bar, pipe) has a length of 50 inches.

The speed of sound in steel is approximately c = 200,000 in/sec.

Now consider that the beam is initially at rest. It then undergoes a free-fall of 18 inches. It is then stopped abruptly by a rigid floor, whereupon it becomes a fixed-free beam.

Also assume 5% damping for each mode.

The first three natural frequencies are

$$f_1 = 1000\ \text{Hz}$$

$$f_3 = 3000\,\text{Hz}$$

$$f_5 = 5000\,\text{Hz}$$

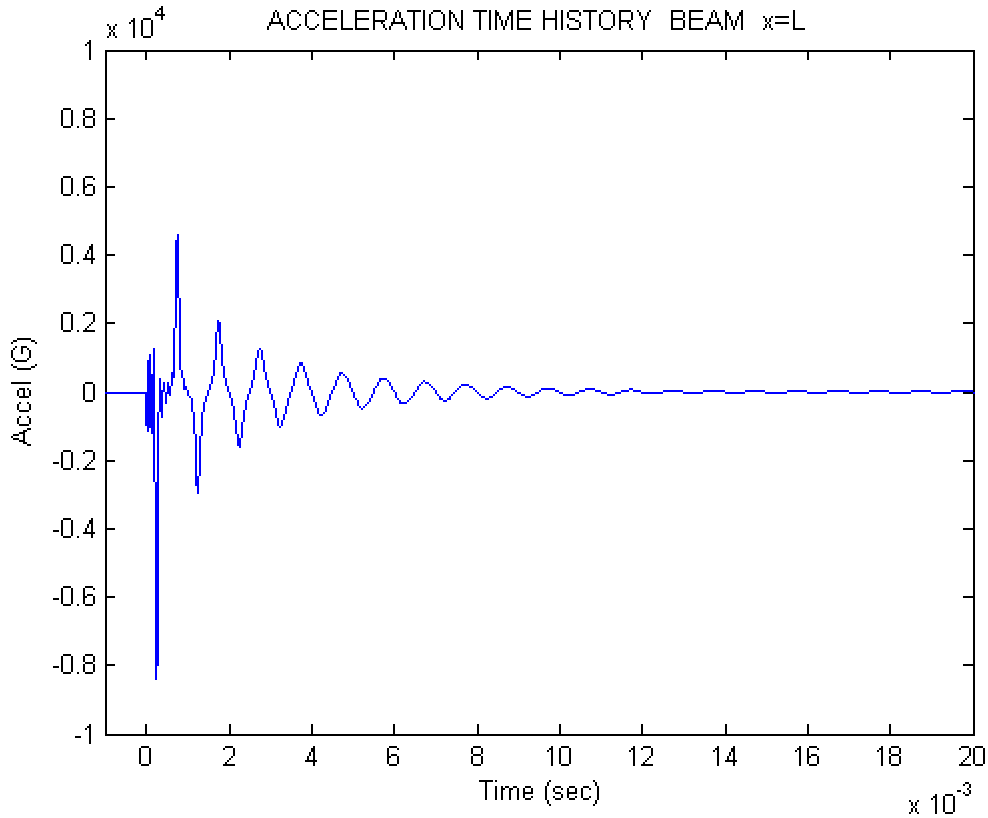


Figure 1.

The resulting acceleration time history is shown in Figure 1 as calculated per equation (17).

Note that this is an idealization. Practical limitations are discussed later.

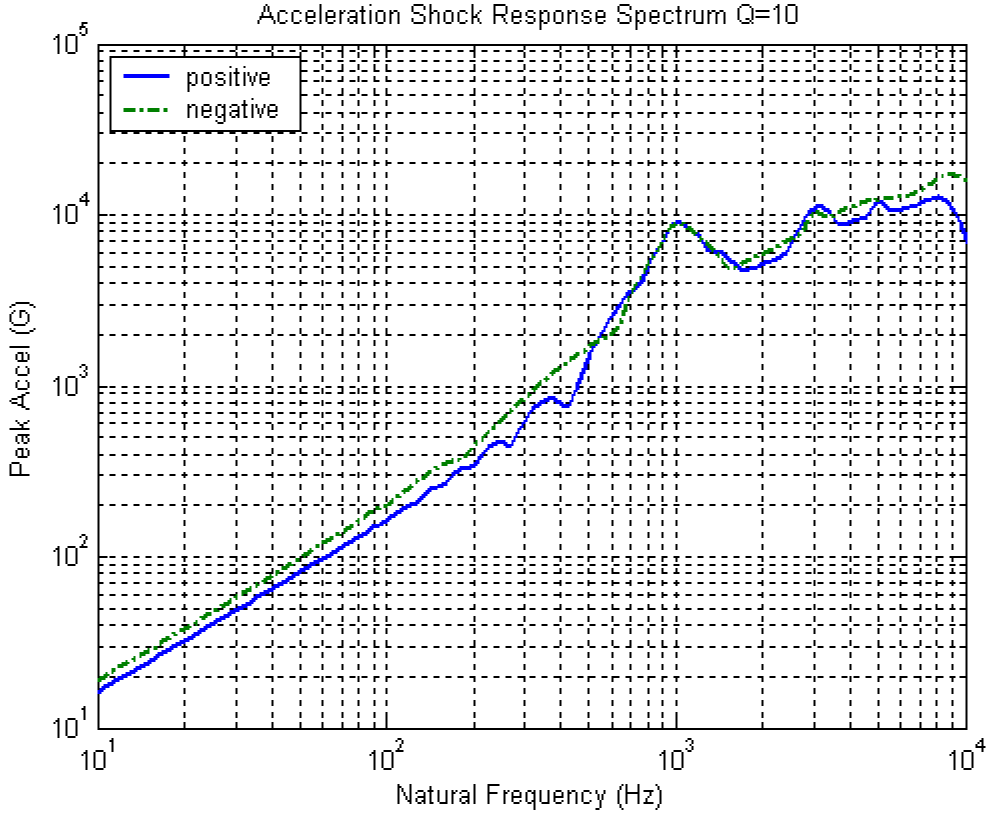


Figure 2.

The resulting SRS is shown in Figure 2.

Larger drop heights would yield higher levels, assuming linearity. The acceleration level is proportional to the square root of the drop height for a given natural frequency.

<u>Sanity Check</u>

Consider a single-degree-of-freedom system dropped from rest. The maximum acceleration from Reference 3 in the time domain is

$$\text{Max Accel} = \omega_n \sqrt{2g\Delta h} \tag{7}$$

Now model the fixed-free beam as a single-degree-of-freedom system.

$$\text{Max Accel} = (1000(2\pi)\ \text{rad/sec})\sqrt{2(386\,\text{in/sec\^{}2})(18\,\text{inch})} \tag{8}$$

$$\text{Max Accel} = 1919\ \text{G} \tag{8}$$

The peak absolute acceleration for the beam in Figure 1 is 8385 G in the time domain. This value is higher because it is the response of several modes and because continuous systems have a gain factor.

Design Concept

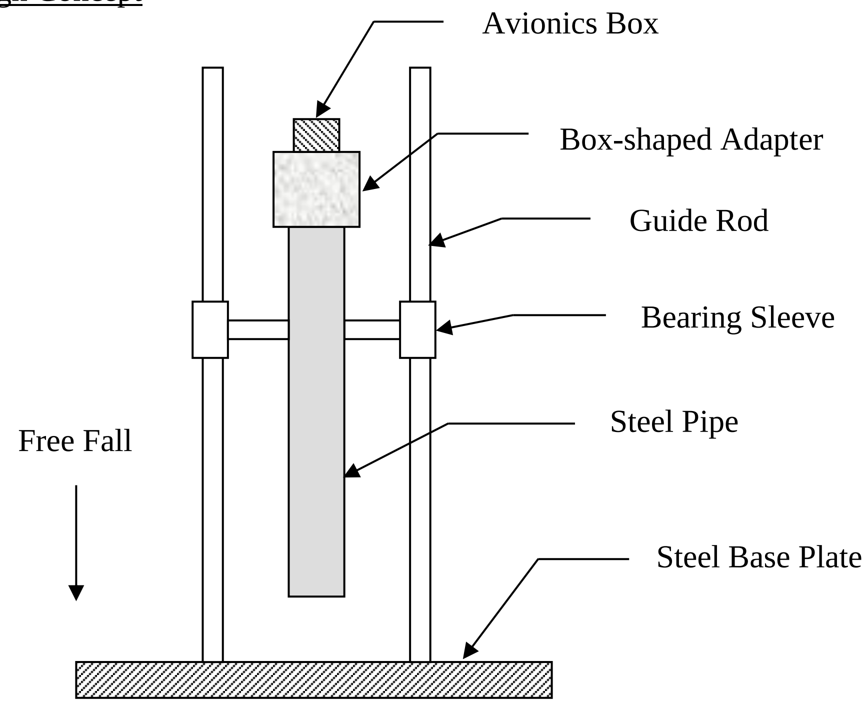


Figure 3.

The steel pipe is "tuned" to the knee frequency of the SRS specification. The mass of the adapter and avionics component must be considered in this calculation.

Furthermore, a mechanism is needed at the base to prevent rebound. This could be some sort of latching mechanism or even an electromagnet.

A mechanism would also be needed to raise the pipe to the desired initial height.

The pipe could potentially be replaced by other cross-sectional shapes.

This method should provide good repeatability. Also, it is one of the few shock test method that can be analyzed mathematically, at least through simple calculations.

## Practical Concerns

Here are some concerns:

1. The floor or base is not perfectly rigid.
2. The guides may reduce the actual initial velocity.
3. Some of the energy is converted into bending modes.
4. Plastic deformation may occur.

Rebound is another concern.

## Concept Test

The basic concept can easily be verified. This can be done by mounting a shock accelerometer on the top end of a rod. The test technician could then hold the rod, vertically, in one hand, a few inches above the ground. He would then let the rod slide through his hand, catching after it impacted the ground. The measured data could then be compared to theory.

## References


1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
2. T. Irvine, Longitudinal Natural Frequencies of Rods, Rev D, Vibrationdata, 2010.
3. T. Irvine, Simple Drop Shock Rev D, Vibrationdata 2004.

APPENDIX A

Internal Force and Normal Stress

Again, the displacement is

$$u(x,t) = \frac{8vL}{\pi^2 c} \sum_{n=1,3,5,\dots}^{\infty} \left\{ \frac{1}{n^2} \sin\left(\frac{\omega_n x}{c}\right) \sin(\omega_n t) \right\} \tag{A-1}$$

$$\omega_n = n\pi \frac{c}{2L} \tag{A-2}$$

The velocity is

$$\dot{u}(x,t) = \frac{8vL}{\pi^2 c} \sum_{n=1,3,5,\dots}^{\infty} \left\{ \frac{\omega_n}{n^2} \sin\left(\frac{\omega_n x}{c}\right) \cos(\omega_n t) \right\} \tag{A-3}$$

$$\dot{u}(x,t) = \frac{8vL}{\pi^2 c} \sum_{n=1,3,5,\dots}^{\infty} \left\{ \frac{1}{n^2} n\pi \frac{c}{2L} \sin\left(\frac{\omega_n x}{c}\right) \cos(\omega_n t) \right\} \tag{A-4}$$

$$\dot{u}(x,t) = \frac{8vL}{\pi^2 c} \pi \frac{c}{2L} \sum_{n=1,3,5,\dots}^{\infty} \left\{ \frac{1}{n} \sin\left(\frac{\omega_n x}{c}\right) \cos(\omega_n t) \right\} \tag{A-5}$$

$$\dot{u}(x,t) = \frac{4v}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \left\{ \frac{1}{n} \sin\left(\frac{\omega_n x}{c}\right) \cos(\omega_n t) \right\} \tag{A-6}$$

The velocity magnitude for mode 1 is

$$|\dot{u}_1(x)| = \frac{4\,v}{\pi}\ \sin\left(\frac{\omega_n x}{c}\right) \tag{A-7}$$

The maximum modal velocity for mode 1 is

$$\dot{u}_{1,max} = \frac{4\,v}{\pi} \tag{A-8}$$

The strain is

$$\frac{\partial}{\partial x}u(x,t) = \frac{8vL}{\pi^2 c^2} \sum_{n=1,3,5,\ldots}^{\infty} \left\{\frac{\omega_n}{n^2}\cos\left(\frac{\omega_n x}{c}\right)\sin(\omega_n t)\right\} \tag{A-9}$$

The normal stress $\sigma$ is

$$\sigma = E\frac{\partial}{\partial x}u(x,t) = \frac{8EvL}{\pi^2 c^2} \sum_{n=1,3,5,\ldots}^{\infty} \left\{\frac{\omega_n}{n^2}\cos\left(\frac{\omega_n x}{c}\right)\sin(\omega_n t)\right\} \tag{A-10}$$

$$\sigma = \frac{8EvL}{\pi^2 c^2} \sum_{n=1,3,5,\ldots}^{\infty} \left\{\frac{1}{n^2}\,n\pi\,\frac{c}{2L}\cos\left(\frac{\omega_n x}{c}\right)\sin(\omega_n t)\right\} \tag{A-11}$$

$$\sigma = \frac{8EvL}{\pi^2 c^2}\,\pi\,\frac{c}{2L} \sum_{n=1,3,5,\ldots}^{\infty} \left\{\frac{1}{n}\cos\left(\frac{\omega_n x}{c}\right)\sin(\omega_n t)\right\} \tag{A-12}$$

$$\sigma = \frac{4\,E\,v}{\pi c} \sum_{n=1,3,5,\ldots}^{\infty} \left\{\frac{1}{n}\cos\left(\frac{\omega_n x}{c}\right)\sin(\omega_n t)\right\} \tag{A-13}$$

The stress magnitude for mode 1 is

$$|\sigma_1(x)| = \frac{4\,E\,v}{\pi c}\cos\left(\frac{\omega_n x}{c}\right) \tag{A-14}$$

The maximum modal stress for mode 1 is

$$\sigma_{1,max} = \frac{4\,E\,v}{\pi c} \tag{A-15}$$

$$\dot{u}_{1,max} = \frac{4\,v}{\pi} \tag{A-16}$$

$$\sigma_{1,max} = \frac{E}{c}\,\dot{u}_{1,max} \tag{A-17}$$

$$c = \sqrt{E/\rho} \tag{A-18}$$

$$E = \rho\,c^2 \tag{A-19}$$

$$\sigma_{1,max} = \rho c\,\dot{u}_{1,max} \tag{A-20}$$

The axial force P is

$$P(x,t) = A\sigma(x,t) = EA\frac{\partial}{\partial x}u(x,t) \tag{A-21}$$

$$P(x,t) = \frac{8EAvL}{\pi^2 c^2}\sum_{n=1,3,5,\ldots}^{\infty}\left\{\frac{\omega_n}{n^2}\cos\left(\frac{\omega_n x}{c}\right)\sin(\omega_n t)\right\} \tag{A-22}$$