

# STATISTICAL DEGREES OF FREEDOM Revision A

By Tom Irvine  
Email: tomirvine@aol.com

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## INTRODUCTION

The power spectral density calculation has associated error sources.

The purpose of this report is to discuss the statistical stability of sampled time history data with respect to power spectral density calculation. The stability is expressed in terms of the statistical degrees of freedom parameter and in terms of *standard error*. Other error sources discussed in this report are *smear, leakage, and bias*.

## POWER SPECTRAL DENSITY BACKGROUND

Power spectral density functions may be calculated via three methods:

1. Measuring the RMS value of the amplitude in successive frequency bands, where the signal in each band has been bandpass filtered.
2. Taking the Fourier transform of the autocorrelation function. This is the Wiener-Khintchine approach.
3. Taking the limit of the Fourier transform  $X(f)$  times its complex conjugate divided by its period  $T$  as the period approaches infinity. Symbolically, the power spectral density function  $X_{\text{PSD}}(f)$  is

$$X_{\text{PSD}}(f) = \lim_{T \rightarrow \infty} \frac{X(f)X^*(f)}{T} \quad (1)$$

These methods are summarized in Reference 1. Only the third method is considered in this report. The following equations are taken from Reference 2.

The discrete Fourier transform is

$$X(m) = \Delta t \sum_{n=0}^{N-1} x(n\Delta t) \exp(-j2\pi m \Delta f n \Delta t) \quad (2)$$

The inverse transform is

$$x(n) = \Delta f \sum_{m=0}^{N-1} X(m\Delta f) \exp(j2\pi m\Delta f n \Delta t) \quad (3)$$

These equations give the Fourier transform values  $X(m)$  at the  $N$  discrete frequencies  $m \Delta f$  and give the time series  $x(n)$  at the  $N$  discrete time points  $n \Delta t$ . The total period of the signal is thus

$$T = N\Delta t \quad (4)$$

where

$N$ =number of samples in the time function and in the Fourier transform

$T$ =record length of the time function

$\Delta t$ =time sample separation

Consider a sine wave with a frequency such that one period is equal to the record length. This frequency is thus the smallest sine wave frequency which can be resolved. This frequency  $\Delta f$  is the inverse of the record length.

$$\Delta f = 1/T \quad (5)$$

This frequency is also the frequency increment for the Fourier transform.

## STATISTICAL DEGREES OF FREEDOM

### Signal Analysis Approach

The statistical degree of freedom parameter is defined from References 3 and 4 as follows:

$$\text{dof} = 2BT \quad (6)$$

where dof is the number of statistical degrees of freedom and  $B$  is the bandwidth of an ideal rectangular filter. This filter is equivalent to taking the time signal “as is,” with no tapering applied. Note that the bandwidth  $B$  equals  $\Delta f$ , again assuming an ideal rectangular filter.

The 2 coefficient in equation (6) appears to result from the fact that a single-sided power spectral density is calculated from a double-sided Fourier transform. The symmetries of the Fourier transform allow this double-sided to single-sided conversion.

For a single time history record, the period is  $T$  and the bandwidth  $B$  is the reciprocal so that the  $BT$  product is unity, which is equal to 2 statistical degrees of freedom from the definition in equation (6).

A given time history is thus worth 2 degrees of freedoms, which is poor accuracy per Chi-Square theory, as well as per experimental data per Reference 3, as shown in Appendix A. Note that the Chi-Square theory is discussed later.

### Breakthrough

The breakthrough is that a given time history record can be subdivided into small records, each yielding 2 degrees of freedom, as discussed in Reference 5 for example. The total degrees of freedom value is then equal to twice the number of individual records. The penalty, however, is that the frequency resolution widens as the record is subdivided. Narrow peaks could thus become *smearred* as the resolution is widened.

An example of this subdivision process is shown in Table 1. The process is summarized in equations (7) through (11).

Table 1. Example: 4096 samples taken over 16 seconds, rectangular filter.					
Number of Records NR	Number of Time Samples per Record	Period of Each Record $T_i$ (sec)	Frequency Resolution $B_i=1/T_i$ (Hz)	dof per Record $=2B_i T_i$	Total dof
1	4096	16.	0.0625	2	2
2	2048	8.	0.125	2	4
4	1024	4.	0.25	2	6
8	512	2.	0.5	2	16
16	256	1.	1.	2	32
32	128	.5	2.	2	64
64	64	.25	4.	2	128

Notes:

1. The subscript “i” is used to denote “individual” in Table 1.
2. The rows in the table could be continued until a single sample per record remained.

Also note that:

$$\text{Total dof} = 2 \text{ NR} \tag{7}$$

$$\text{NR} = T / T_i \tag{8}$$

$$B_i = 1 / T_i \tag{9}$$

$$\text{NR} = B_i T \tag{10}$$

$$\text{Total dof} = 2 B_i T \quad (11)$$

## CHI-SQUARE THEORY

### Assumptions

The Chi-Square distribution is a distribution of squared values. It is thus particularly suited to the estimation of the confidence interval of squared functions. This distribution is suited for stationary, random signals. It does not apply to stationary, deterministic signals or to non-stationary signals.

A stationary signal is a signal for which the overall level and the spectral distribution do not vary with time. A deterministic signal is a signal such as a sinusoid which can be completely described by a time-domain equation.

### Signal Analysis Approach

Reference 4 gives the Chi-Square relationship as

$$\frac{s^2}{\sigma^2} = \frac{\chi^2}{\text{dof}} \quad (12)$$

Now consider the mean square value  $s^2$  of a signal measured for a time  $T$ . The true mean square amplitude  $\sigma^2$  will lie within a confidence interval determined by

$$\sigma^2 = \frac{\text{dof}}{\chi^2} s^2 \quad (13)$$

Reference 4 is a user's *manual* approach to signal analysis.

### Textbook Approach

On the other hand, Reference 5 gives a textbook approach as follows.

The statistical degrees of freedom parameter arises from a Chi-Square distribution. Consider a signal with a Gaussian probability density function and a true mean square amplitude of  $\sigma^2$ . The Chi-Square theorem states:

If  $s^2$  is the variance of a random sample of size  $\hat{n}$  taken from a normal population have the true mean square amplitude  $\sigma^2$ , then the random variable

$$X^2 = \frac{(\hat{n} - 1)s^2}{\sigma^2} \quad (14)$$

has a Chi-Square distribution with  $\nu$  degrees of freedom, where  $\nu = \hat{n} - 1$ .

The values of the random variable  $X^2$  are calculate from each sample by the formula

$$\chi^2 = \frac{(\hat{n} - 1)s^2}{\sigma^2} \quad (15)$$

$$\chi^2 = \frac{\nu s^2}{\sigma^2} \quad (16)$$

Note that the variance is also called the mean square value.

### Reconciliation

A bridge is needed between statistical theory presented in signal processing manuals and that given in textbooks.

For signal processing, the  $\hat{n}$  term in equations (14) and (15) does not represent the number of time history samples. Rather,  $\hat{n}$  represents twice the number of non-overlapping power spectral density records NR, each with period  $T_i$ .

Thus, with respect to equation (16),

$$\nu = \text{Total dof} = 2 \text{ NR} \quad (17)$$

$$\nu = 2 B_i T \quad (18)$$

The subscript is used to denote that B is the reciprocal of the individual record time period rather than the total time period T.

In summary, the number of degrees of freedom can be increased by widening the frequency resolution.

The textbook approach from Reference 5 can thus be meshed with the user's manual approach from Reference 4, by setting  $\nu = \text{dof}$ .

### Confidence Interval Example

Consider three of the cases from the previous example, summarized in Table 1. Determine the bounds on the measured mean square value  $S^2$  for a 95% confidence interval. The results are shown in Table 2, where the calculation is carried out using the values <sup>1</sup> in Reference 5.

Total dof	Bounds on $S^2$ for 95% Confidence	Standard Error
8	0.27 to 2.19	-
16	0.41 to 1.80	-
32	0.57 to 1.55	0.250
64	0.68 to 1.37	0.177

The point of the data in Table 2 is that the boundaries narrow as the number of degrees-of-freedom is increased, for a given confidence percentage. The measured mean square value is thus more likely to represent the true value for a higher number of degrees-of-freedom.

Note that the Chi-Square distribution approximates a Gaussian distribution for large degree-of-freedom values. Reference 3 gives a practical lower limit value of 30 degrees to make this approximation. In this case, the approximate standard error  $\epsilon_0$  is given by

$$\epsilon_0 = \frac{1}{\sqrt{B_i T}} \quad (19)$$

$$\epsilon_0 = \frac{1}{\sqrt{\frac{1}{2}(\text{Total dof})}} \quad (20)$$

The standard error values are given in Table 2 for the cases with at least 30 degrees-of-freedom. Note that the standard error represents the true standard deviation relative to the measured value. Thus, the true standard deviation has bounds of  $\pm 17.7\%$  for the case of 64 degrees-of-freedom, per the Gaussian approximation, as shown in Table 2.

Note that standard deviation refers to the standard deviation of the individual power spectral density points in this context. Standard deviation can also refer to the standard deviation of the time history points in another context.

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<sup>1</sup> A Fortran program was also written to carry out this calculation since the tabular Chi-Square values in Reference 5 were only given up to 30 degrees-of-freedom.

## FURTHER PROCESSING CONCEPTS

### Window

A window is typically applied to each time segment during the power spectral density calculation, as discussed in Reference 3, 4, and 6. The purpose of the window is to reduce a type of error called *leakage*. One of the most common windows is the Hanning window, or the cosine squared window. This window tapers the data so that the amplitude envelope decreases to zero at both the beginning and end of the time segment. The Hanning window  $w(t)$  can be defined as

$$w(t) = \begin{cases} 1 - \cos^2\left[\pi\frac{t}{T}\right], & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases} \quad (21)$$

The window operation reduces the *leakage error* but also has the effect of increasing the effective bandwidth  $B$ . The number of statistical degrees-of-freedom is thus reduced. The boundaries associated with the confidence intervals thus widen.

Also, a normalization factor of  $\sqrt{8/3}$  is applied to the Hanned data to compensate for the lost energy, from Reference 7.

### Overlap

The lost degrees-of-freedom can be recovered by overlapping the time segments, each of which is subjected to a Hanning window. Nearly 90% of the degrees-of-freedom are recovered with a 50% overlap, according to Reference 3.

The concept of windows and overlapping is represented in Figure 1.

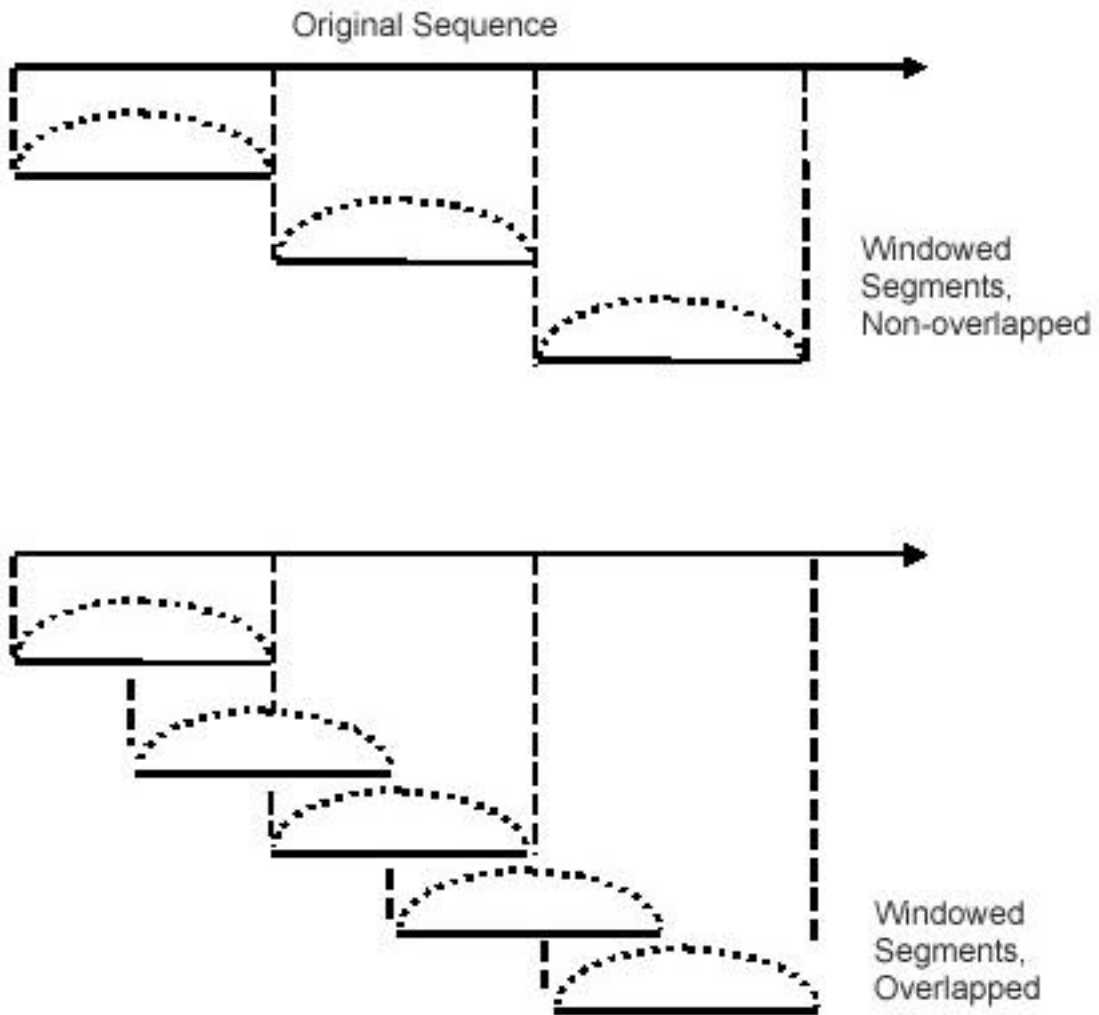


Figure 1.

Fast Fourier Transform

The discrete Fourier transform was given in equation (2). The solution to this equation requires a great deal of processing steps for a given time history. Fast Fourier transform methods have been developed, however, to greatly reduce the required steps. These methods typically require that the number of time history data points be equal to  $2^N$ , where N is some integer. The derivation method is via a butterfly algorithm, as shown, for example, in Reference 8.



Records with sample numbers which are not equal to an integer power of 2 can still be processed via the fast Fourier transform method. Such a record must either be truncated or padded with zeroes so that its length becomes an integer power of 2.

Padding with zeroes obviously underestimates the true power spectral density amplitudes. A compensation scaling factor can be applied to account for this, however.

Truncation, on the other hand, sacrifices some of the original signal information. Nevertheless, most of the original signal can still be used by subdividing the data and by using overlap techniques.

### INTERMEDIATE SUMMARY

Time history data is subdivided into segments to increase the statistical-degrees-of-freedom by broadening the frequency bandwidth. Next, a window is applied to each segment to taper the ends of the data. Finally, overlapping is used to recover degrees-of-freedom lost during the window operations. The effect of these steps is to increase the accuracy of the power spectral density data. Nevertheless, there are some tradeoffs as shown in the following examples.

### EXAMPLES

#### Example 1: Fixed Bandwidth with Variable Duration

Consider that an engineer is planning a modal test. A modal shaker will be used to apply random vibration to a launch vehicle fairing. The engineer needs to determine the duration for each data run.

The engineer could first choose the required resolution bandwidth and then determine the desired accuracy. These parameters would then dictate a required duration.

For example, consider that a spectral bandwidth of 2 Hz is required. This corresponds to a segment duration of 0.5 seconds. The results of this choice are shown in Table 3, where the total duration is calculated for several degree cases.

Table 3. Parameters for $B_i=2$ Hz Example, Rectangular Filter					
Total dof	Bounds on $S^2$ for 95% Confidence	Standard Error	Segment Duration (sec)	Number of Segments	Total Duration (sec)
16	0.41 to 1.80	-	0.5	8	4
32	0.57 to 1.55	0.250	0.5	16	8
64	0.68 to 1.37	0.177	0.5	32	16
128	0.77 to 1.20	0.125	0.5	64	32

Thus, the standard error is 12.5% for the 128 dof case. This means that the confidence value is 95% that the measured standard deviation value is within  $\pm 12.5\%$  of the true value. Again, this assumes a stationary, random signal.

The 128 dof case is the best choice in this case. Higher dof could be achieved with a longer duration, but the accuracy gain would be a diminishing returns effect.

The parameters in Table 3 assume a rectangular filter for simplicity. In reality, a Hanning window with 50% overlap would probably be used.

Example 2: Fixed Bandwidth and Fixed Duration

An engineer must calculate a power spectral density function for a flight time history. The required spectral bandwidth is 5 Hz. The data is stationary, random vibration as shown in Figure 2. The duration is 8 seconds. Note that the *flight data* in Figure 2 is actually synthesized data; with an amplitude of approximately  $1.0 G^2/Hz$  over the frequency range 10 Hz to 200 Hz. The overall level is 15.0 GRMS.

The engineer must choose whether to represent the data in terms of a single segment or in terms of a family of segments. Note that the term *segment* is now being used in a new context. The data would be further broken into sub-segments to form the average for each final segment. Also note that the dof per segment values are not additive since the final segments are represented individually rather than averaged.

Table 4. Parameters for $B_i=5$ Hz Example, 8 Second Duration, Rectangular Filter			
Number of Final Segments	Duration of each Final Segment (sec)	dof per Segment	Standard Error per Segment
1	8	80	0.158
2	4	40	0.224
4	2	20	-
8	1	10	-

Recall that a minimum of 30 degrees of freedom is needed to apply the standard error theory.

The power spectral density for the single-segment choice is shown in Figure 3. The mean amplitude of the power spectral density points in Figure 3 is  $1.0 G^2/Hz$ . Furthermore, the standard deviation is 0.065, which is actually better than the theoretical standard error value of 0.158. Note that the ideal standard deviation for this synthesized case is zero.

All four processing choices are shown in Figure 4. Obviously, the price of additional segment curves is less accuracy.

The 1 to 2 second curve from the 8-segment choice is shown in Figure 5. The mean value is fine but the standard deviation has climbed to 0.358. Note the peak at 127 Hz, which

reaches an amplitude of  $2.0 \text{ G}^2/\text{Hz}$ . This amplitude is 3 dB higher than the synthesized amplitude. The curve in Figure 4 could lead to an erroneous conclusion that the flight vehicle has a 127 Hz forcing frequency or natural frequency.

Finally, the maximum envelope for the 8-segment choice is shown in Figure 6. This maximum envelope clearly overestimates the true amplitude which is  $1.0 \text{ G}^2/\text{Hz}$ .

Thus, the correct choice for this example would be a single 8-second curve, as shown in Figure 3.

### Example 3: Non-stationary Data

Now repeat example 2 for the time history in Figure 7. This is the same time history from Example 3 except that a 20 Hz, 30 G signal is superimposed during the first one-second interval.

The family of power spectral density curves is shown in Figure 8. Note that the scaling of the Y-axis was greatly increased to accommodate the 20 Hz spectral peak, relative to the scaling in Figure 4. In particular, note that the peak amplitude increases as shorter segments are taken. In other words, a segment duration greater than one second obfuscates the true amplitude of the 20 Hz spectral peak. This error is called *bias error*.

*Bias error* has a similar effect as *smear error*. The difference is that bias error results from lengthening the final segment duration, where as *smear error* results from broadening the frequency resolution.

An individual power spectral density curve taken from 0 to 1 second is given in Figure 9. This curve is the first from the 8-segment case. The overall level is 25.7 GRMS which is close to the expected value of 26.2 GRMS.

Thus, the correct choice for this example would be to represent the data in terms of one-seconds segments.

## DERIVATION OF TEST SPECIFICATIONS FROM FLIGHT DATA

Some vibration data is reasonably stationary. Real flight data has a non-stationary tendency, however.

The need for accuracy requires increased segment duration, to minimize *standard error*. On the other hand, a longer segment duration could obfuscate any transient responses, resulting in *bias error*.

MIL-STD-1540C, paragraph 3.3.4, is very concerned with *bias error*. This standard thus recommends a one-second segment duration for flight data. Unfortunately, this standard fails to address the *standard error* (no pun intended) which results from using brief segments.

The proper choice of segment duration for non-stationary flight data remains a dilemma. The real problem is that a stationary random vibration test specification is sought to represent a non-stationary flight environment.

This dilemma can be partially solved by separating transient effects from steady-state vibration. In other words, multiple specifications can be derived to envelop various flight events. Currently, the common practice is to consider some events as shock events and others as steady-state random vibration. Nevertheless, certain events fall somewhere in between these two categories.

Creative but reasonable engineering judgment is thus needed to derive test specifications to cover real flight data. For example, the data in Figure 7 could be represented in terms of a sine-on-random specification. In addition, the effect of the 20 Hz sinusoid on a payload could be analyzed in a coupled-loads model.

Now consider that obvious transient events have been removed from the flight data. A better method of characterizing the remaining flight data and would be to use a histogram approach. This will be the subject of a future report.

## CONCLUSION

Engineers use the power spectral density function to represent a signal by a series of sinusoids at discrete frequencies. This process has associated *standard error* and *bias error*. *Smear error* and *leakage error* are also considerations. Engineers must understand these error sources in order to properly interpret and utilize power spectral density data.

## REFERENCES

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### SYNTHESIZED RANDOM VIBRATION

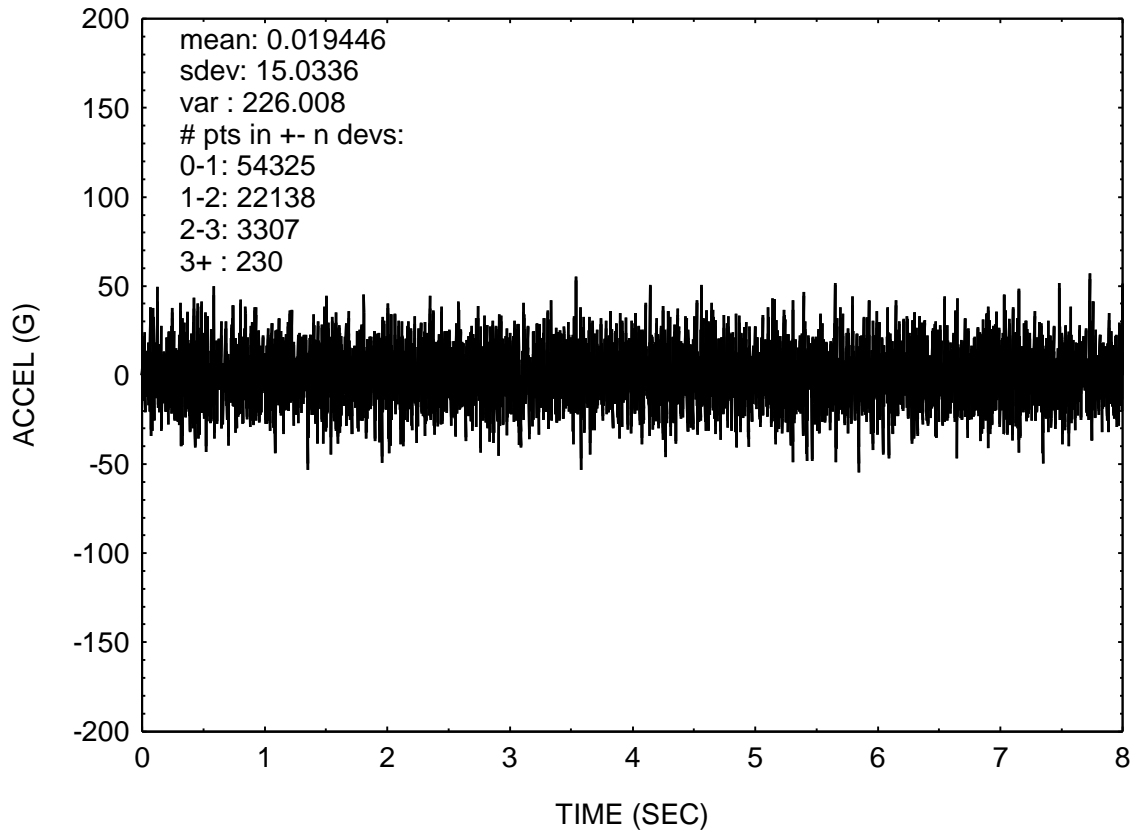


Figure 2.

POWER SPECTRAL DENSITY OF SYNTHESIZED RANDOM VIBRATION  
0 TO 8 SEC  $\Delta f = 4.88$  Hz 80 DEGREES OF FREEDOM

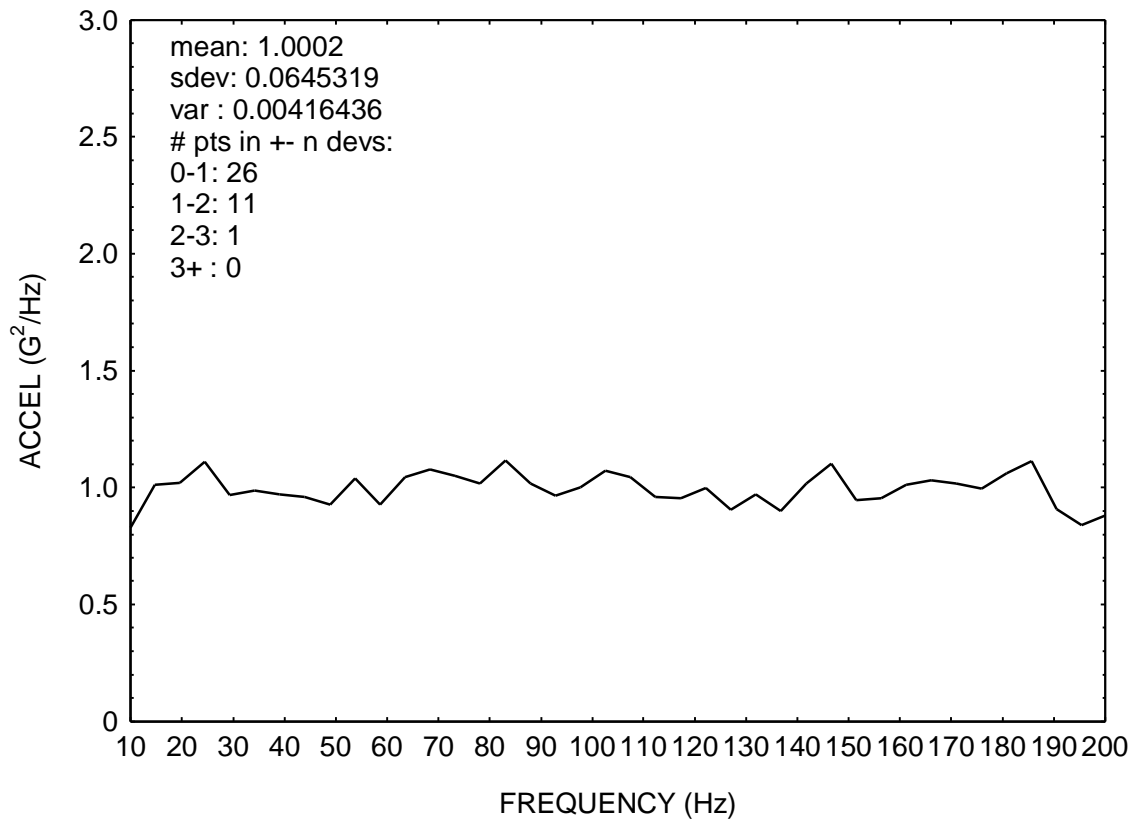


Figure 3.

POWER SPECTRAL DENSITY OF SYNTHESIZED RANDOM VIBRATION  
0 TO 8 SEC  $\Delta f = 4.88$  Hz

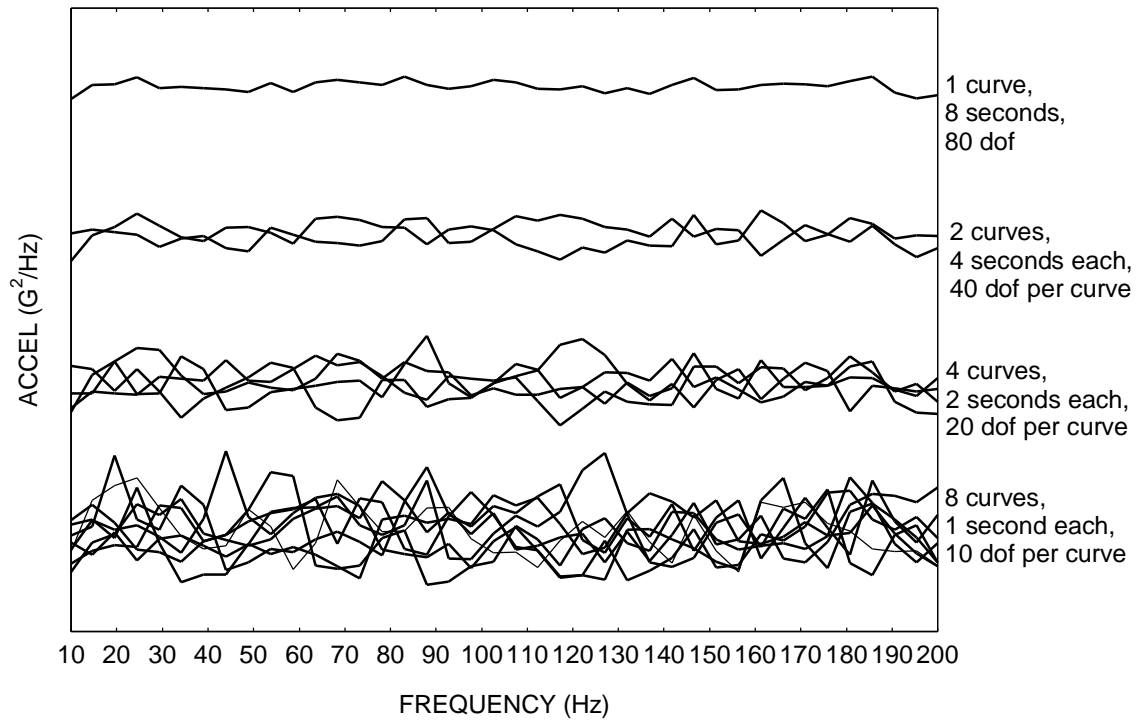


Figure 4.

POWER SPECTRAL DENSITY OF SYNTHESIZED RANDOM VIBRATION  
1 TO 2 SEC  $\Delta f = 4.88$  Hz 10 DEGREES OF FREEDOM

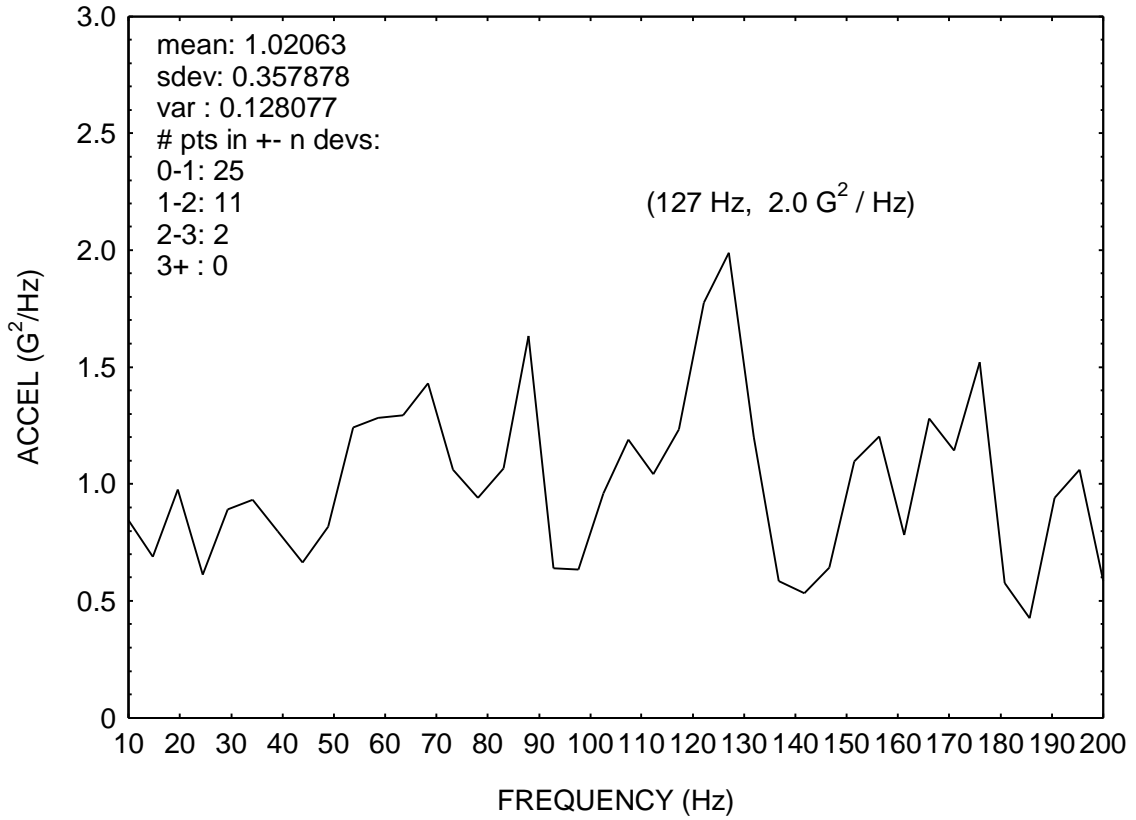


Figure 5.



POWER SPECTRAL DENSITY OF SYNTHESIZED RANDOM VIBRATION  
0 TO 8 SEC  $\Delta f = 4.88$  Hz

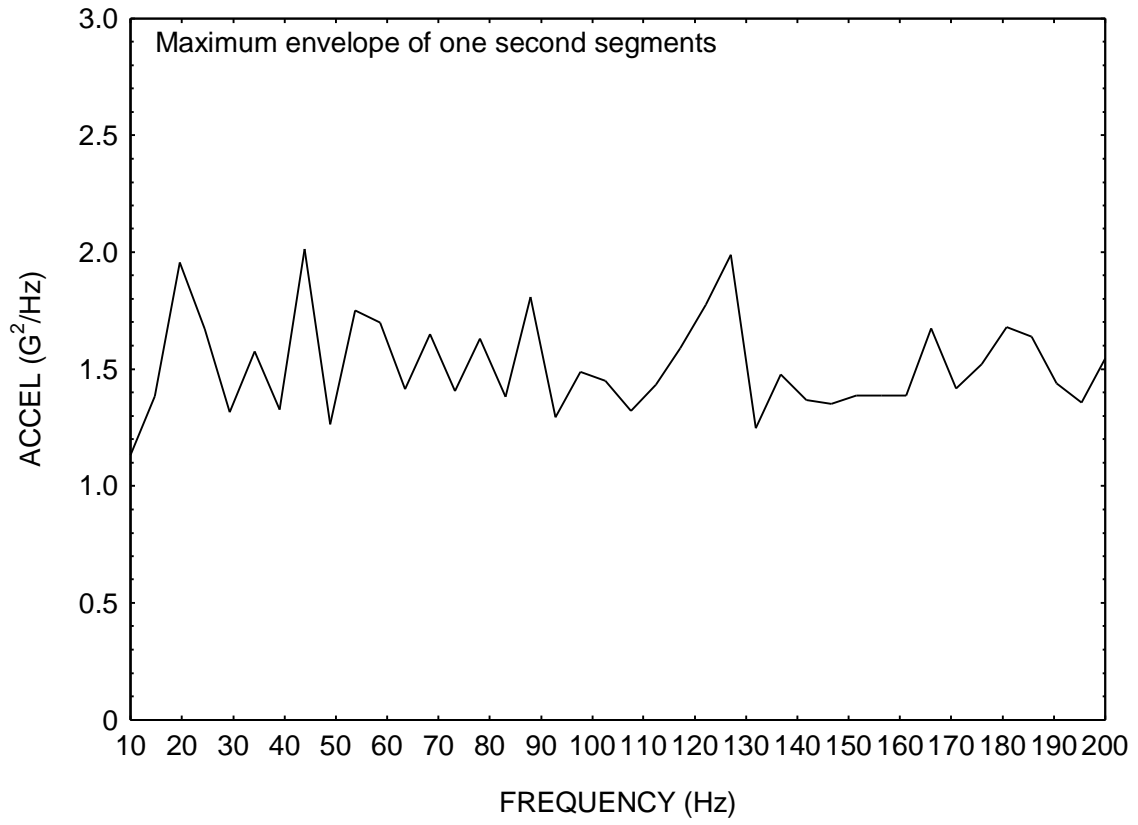


Figure 6.

SYNTHESIZED RANDOM VIBRATION ( NON-STATIONARY EXAMPLE )

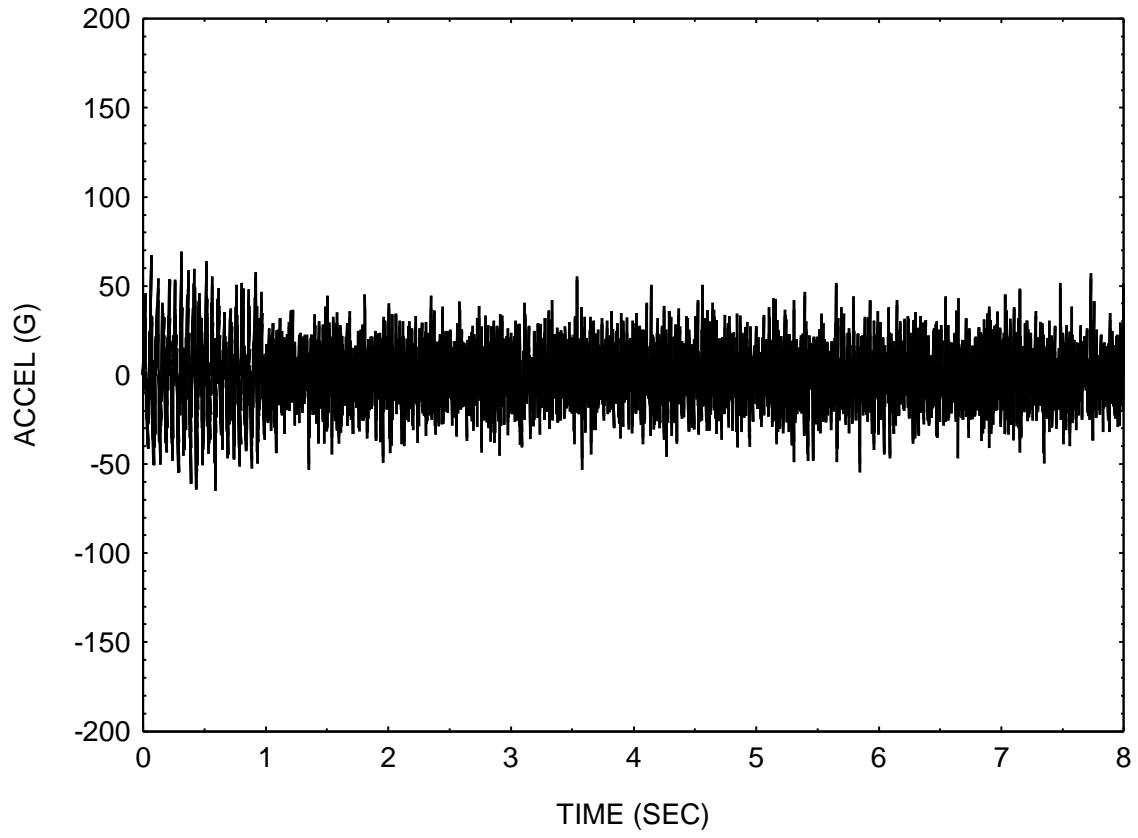


Figure 7.

POWER SPECTRAL DENSITY  
SYNTHESIZED RANDOM VIBRATION (NON-STATIONARY EXAMPLE)  
0 TO 8 SEC  $\Delta f = 4.88$  Hz

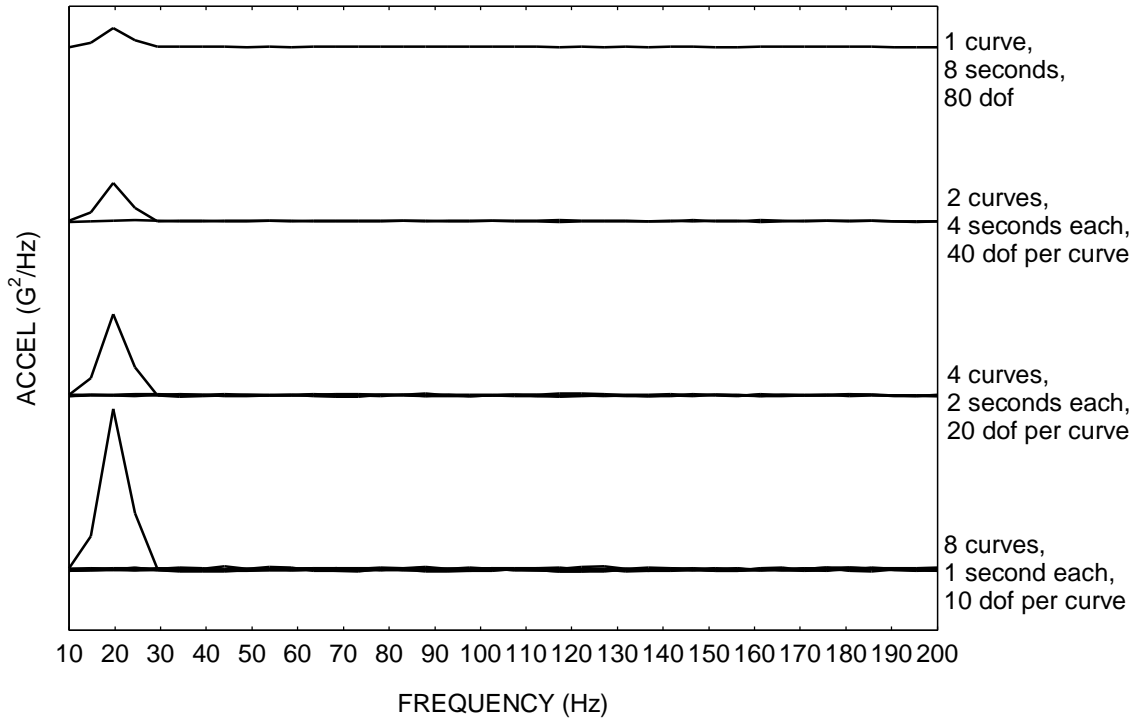


Figure 8.

POWER SPECTRAL DENSITY  
SYNTHESIZED RANDOM VIBRATION (NON-STATIONARY EXAMPLE)  
0 TO 1 SEC  $\Delta f = 4.88$  Hz 10 DEGREES OF FREEDOM

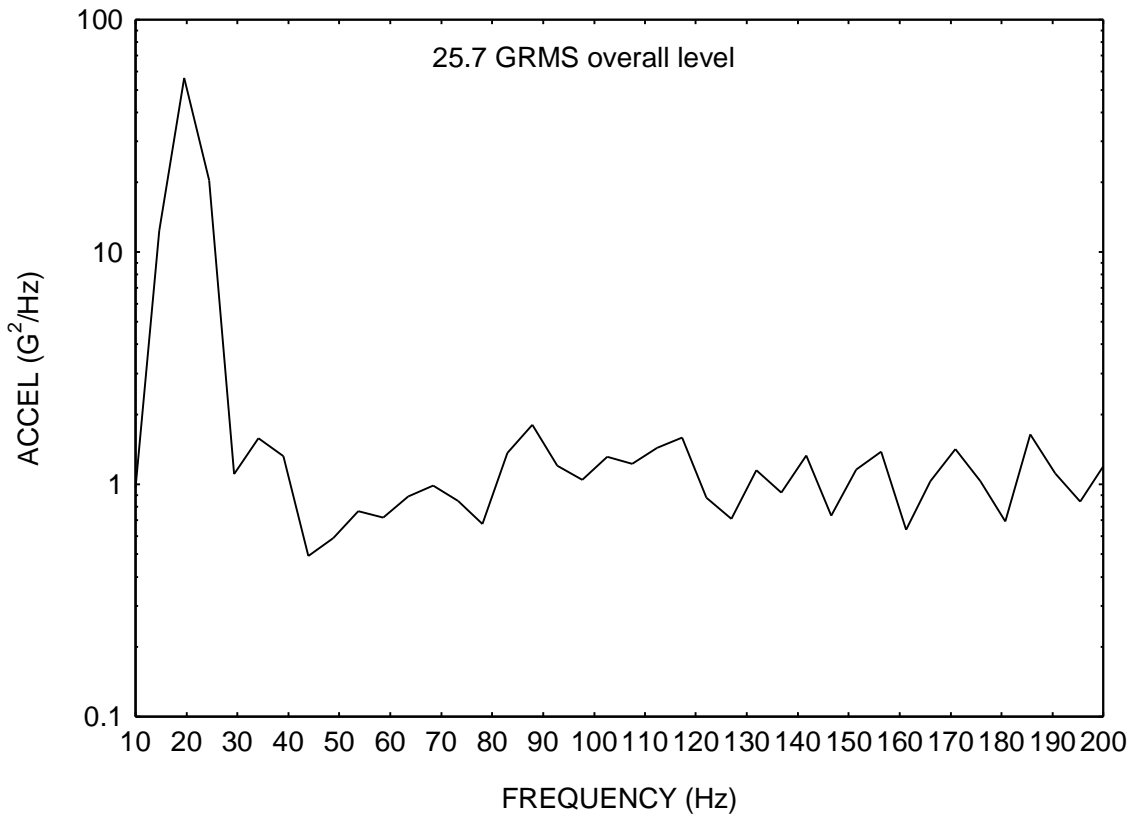


Figure 9.

## APPENDIX A

### **6.1.6 Statistical Stability**

Two parameters affect statistical stability:

- Resolution bandwidth,  $B$ .
- Record length  $T = N \Delta t$

**HOW TO COMPUTE PSDs**

**6 - 8**

**MAC/RAN Applications**

We have chosen  $f_s = 2\text{sps}$  so that  $G_{xxk} = 1$  for the example. The d.f. is  $n = 2$  in the first plot,  $n = 20$  in the second, 200 in the third, and 2000 in the fourth. The standard errors are:

Figure	d.f.	$\epsilon_0 = 1/\sqrt{B_s T}$
6-3a	2	1.00
6-3b	20	.30
6-3c	200	.10
6-3d	2000	.03

which are reflected in the fluctuations of the spectra in Figure 6-3.

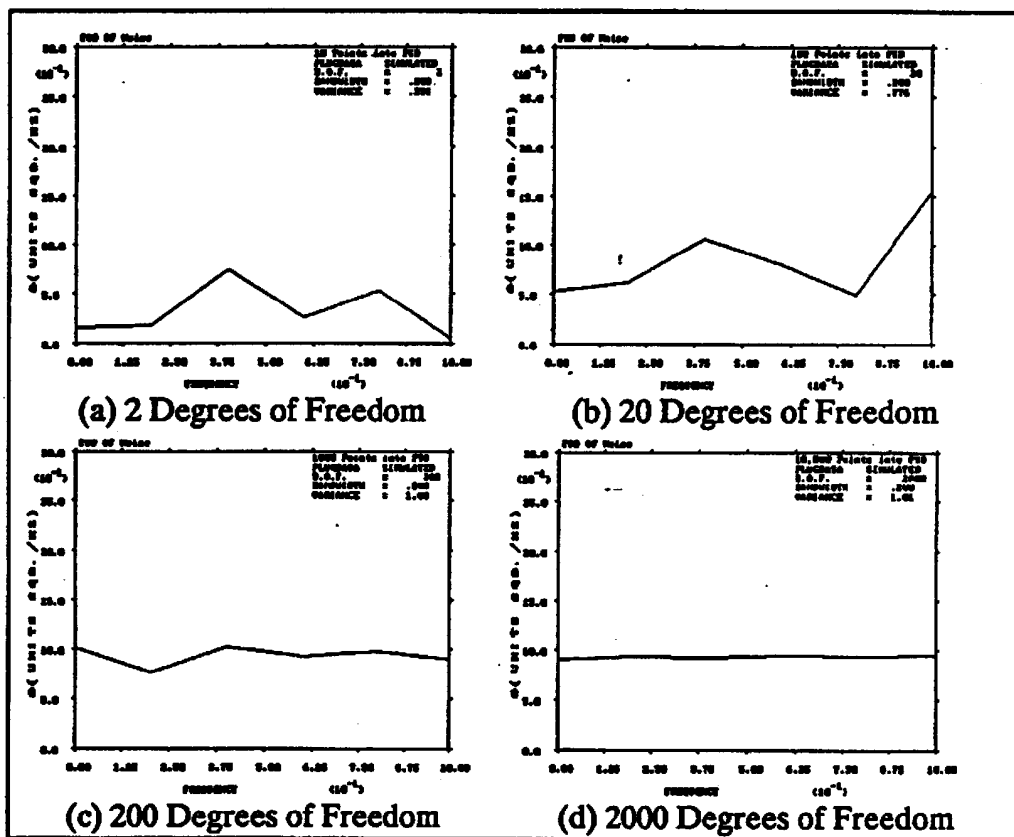


Figure 6-3. Example of Statistical Stability

Power spectrum confidence limits are not computed in MAC/RAN but are easily determined by reference to either  $X^2$  distributed tables or Gaussian distribution tables depending on  $n$ .

It can be shown, that under reasonable conditions, PSD estimates approximately follow a chi-square ( $\chi^2$ ) distribution with  $n$  d.f. where

$$n = 2B_k T \quad (6.5)$$

Furthermore, for  $n$  moderately large, say  $n \geq 30$ , the  $\chi^2$  distribution approximates a Gaussian (normal) distribution. In this case, the normalized standard deviation (standard deviation relative to the value being estimated, i.e., percentage error) is given by

$$\epsilon_0 = \frac{1}{\sqrt{B_k T}} \quad (6.6)$$

The quantity  $\epsilon_0$  is referred to as the "standard error." Proofs and additional details may be found in DTSA, Chapter 5.

The d.f. and associated standard error for the three methods are shown in Table 6-1 which is Table 5-3 in DTSA.

Table 6-1. Degrees of Freedom

PSD Type	d.f. $n = 2B_k T$	$\epsilon_0 = \sqrt{1/B_k T}$
B-T Correlation	$n = 2N/m$	$\sqrt{m/N}$
FFT	$n = 2M$	$\sqrt{1/M}$
Filter	$n = 2B_k T$	$\sqrt{1/B_k T}$

$N$  = Total number of data points

$m$  = Maximum lag value in correlation function

$M$  = Number of complex Fourier coefficients averaged

$B_k$  = HPP bandwidth of numerical filter

$T = N \Delta t$  = record length

The effect of increasing the d.f. is illustrated in Figure 6-3. All four plots are PSDs of white noise of unit variance

$$\sigma^2 = 1$$

so the true spectrum is

$$G_{xxk} = 1/f_N = 2/f_s = 2\Delta t \quad (6.7)$$