SINE SWEEP FREQUENCY PARAMETERS

By
Tom Irvine
Email: tomirvine@aol.com
October 28, 1998

INTRODUCTION

This tutorial gives the derivation of a time history which corresponds to a sine sweep specification. The tutorial is specifically concerned with the frequency parameter, which is a function of time. Surprisingly, the frequency function for the spectral domain differs from that of the sinusoidal argument in the time domain.

Note that the spectral frequency is that which would be seen on an "instantaneous" Fourier transform of the sinusoidal signal.[1]

The purpose of this tutorial is to derive the respective frequency functions. Both linear and logarithmic cases are addressed. Thus, a total of four frequency functions are derived.

The key parameter in the derivation is the total accumulated cycles.

The amplitude is taken as unity for simplicity. Note that the amplitude calculation is much simpler than the frequency calculation.

LINEAR SWEEP RATE

The variables are given in Table 1.

Table 1. Variable Description	
Term	Description
f_1	starting frequency
f_2	ending frequency
t_1	starting time
t_2	ending time
t	elapsed time (independent variable)

[1] This is a bit of an oxymoron since a Fourier transform is calculated over a duration.

Table 1. Variable Description (continued)	
Term	Description
$f_{spectral}$	frequency in the spectral domain
f_{arg}	frequency in the time-domain sine argument
C	accumulated cycles

Assume that all frequency terms are in units of Hertz and that all time terms are in units of seconds.

The goal is to derive an equation for the amplitude Y(t).

$$Y(t) = \sin\left\{2\pi\left[\ f_{arg}(t)\ \right]t\right\} \tag{1a}$$

Note that equation (1a) assumes a unity amplitude for simplicity.
A more fundamental representation of this equation is

$$Y(t) = \sin\left\{2\pi\left[\ C(t)\ \right]\right\} \tag{1b}$$

The spectral frequency function is simply

$$f_{spectral}(t) = \left[\frac{f_2 - f_1}{t_2 - t_1}\right]t + f_1 \tag{2}$$

The accumulate cycle function is found through integration,

$$C(t) = \int_{t_1}^{t} \left\{\left[\frac{f_2 - f_1}{t_2 - t_1}\right]\hat{t} + f_1\right\} d\hat{t} \tag{3}$$

$$C(t) = \left\{\left[\frac{1}{2}\right]\left[\frac{f_2 - f_1}{t_2 - t_1}\right]\left[\hat{t}^{\,2}\right] + f_1\,\hat{t}\right\}\Bigg|_{t1}^{t} \tag{4}$$

$$C(t) = \left[\frac{1}{2}\right]\left[\frac{f_2 - f_1}{t_2 - t_1}\right]\left[t^2 - t_1{}^2\right] + f_1\left[t - t_1\right] \tag{5}$$

Let $t_1 = 0$,

$$C(t) = \left[\frac{1}{2}\right]\left[\frac{f_2 - f_1}{t_2}\right]\left[t^2\right] + f_1\left[t\right] \tag{6}$$

The time-domain sine term requires that

$$f_{arg}(t) = \left[C(t)\right] / \left[\,t\,\right] \tag{7}$$

Substitute equation (6) into (7).

$$f_{arg}(t) = \left\{\left[\frac{1}{2}\right]\left[\frac{f_2 - f_1}{t_2}\right]\left[t^2\right] + f_1\left[t\right]\right\} / \left[\,t\,\right] \tag{8}$$

$$f_{arg}(t) = \left\{\left[\frac{1}{2}\right]\left[\frac{f_2 - f_1}{t_2}\right]\left[t\right] + f_1\right\} \tag{9}$$

Recall equation (2) and apply $t_1 = 0$,

$$f_{spectral}(t) = \left[\frac{f_2 - f_1}{t_2}\right]t + f_1 \tag{10}$$

Note the following relationship for the typical case where the ending frequency is greater than the starting frequency.

$$f_{spectral}(t) > f_{arg}(t) \tag{11}$$

Finally,

$$Y(t) = \sin\left\{2\pi\left\{\left[\frac{1}{2}\right]\left[\frac{f_2 - f_1}{t_2}\right][t] + f_1\right\}t\right\} \tag{12}$$

LOGARITHMIC SWEEP RATE

The number of octaves N is calculated in terms of natural log functions as follows,

$$N = \frac{\ln\left[\frac{f_2}{f_1}\right]}{\ln[2]} \tag{13}$$

The sweep rate R in terms of octaves per time is

$$R = \frac{N}{t_2 - t_1} \tag{14}$$

The spectral frequency function is

$$f_{spectral}(t) = [f_1]\left[2^{R(t - t_1)}\right] \tag{15}$$

The cycle equation is

$$C(t) = \int_{t_1}^{t}\left\{[f_1]\left[2^{R(\hat{t} - t_1)}\right]\right\}d\hat{t} \tag{16}$$

The integration can be carried out per Reference (1). The result is

$$C(t) = \left.\frac{\left\{[f_1]\left[2^{R(\hat{t} - t_1)}\right]\right\}}{R\ln[2]}\right|_{t_1}^{t} \tag{17}$$

Note that the lower limit yields a non-zero term regardless of the limit value.

$$C(t) = \frac{\left\{[f_1]\left[2^{R(t-t_1)}\right]\right\}}{R\ln[2]} - \frac{\{f_1\}}{R\ln[2]} \tag{18}$$

$$C(t) = \frac{\left\{[f_1]\left[-1+2^{R(t-t_1)}\right]\right\}}{R\ln[2]} \tag{19}$$

Now apply $t_1 = 0$,

$$C(t) = \frac{\left\{[f_1]\left[-1+2^{Rt}\right]\right\}}{R\ln[2]} \tag{20}$$

The argument frequency is thus

$$f_{arg}(t) = \frac{\left\{[f_1]\left[-1+2^{Rt}\right]\right\}}{R\,t\,\ln[2]} \tag{21}$$

Again, the following relationship holds for the typical case where the ending frequency is greater than the starting frequency.

$$f_{spectral}(t) > f_{arg(t)} \tag{22}$$

Finally,

$$Y(t) = \sin\left\{ 2\pi\left\{ \frac{[f_1]\left[-1+2^{Rt}\right]}{R\ln[2]} \right\}\right\} \tag{23}$$

REFERENCE

1. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.