LINEAR BUCKLING ANALYSIS USING AN EIGENVALUE SOLUTION

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Introduction

The critical load is the maximum load that a structure can support prior to structural instability or collapse. The collapse of the structure is reached when the displacements become relatively large for a small load increment. Thus, the overall stiffness of the structure becomes very small at the critical load.

Another perspective is that buckling occurs when a structure converts membrane strain energy into bending strain energy.

The buckling problem can be represented in matrix form by the generalized eigenvalue problem.

$$[K_l + \lambda_i K_s][\phi_i] = 0 \tag{1}$$

where

K_l is the global linear stiffness matrix

K_s is the global differential or initial stress stiffness matrix

λ_i are the eigenvalues that when multiplied by the applied loading P gives the critical loading P_{cr}

ϕ_i are the eigenvectors that represent the buckled mode shapes

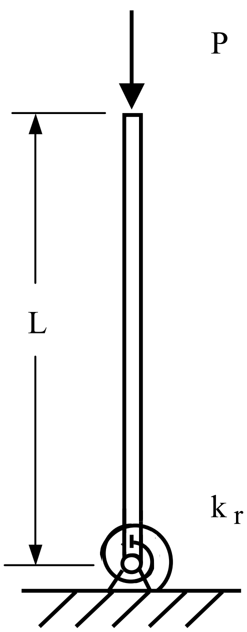
Again,

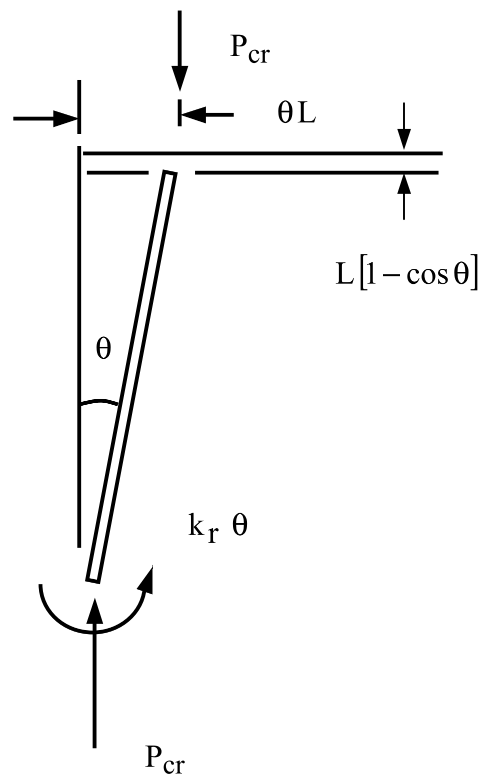
$$P_{cr} = \lambda_i P \tag{2}$$

The linear stiffness matrix K_l is matrix that accounts for the elastic strain energy of the system.

The differential stiffness matrix K_S can also be described as the matrix that accounts for the effect that the applied load P has on the stiffness of the flexural member. It is also the matrix that accounts for the work applied against the system.

Example 1: Rigid Column, Hinged at Base, with Rotational Spring





The total potential energy U is

$$U = \frac{1}{2} k_r \theta^2 - P_{cr}\, L[1 - \cos\theta] \tag{1.1}$$

Note that for a small angle

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \tag{1.2}$$

$$U = \frac{1}{2} k_r \theta^2 - P_{cr}\, L\left[1 - \frac{\theta^2}{2}\right] \tag{1.3}$$

Apply the principle of stationary potential energy.

$$\frac{dU}{d\theta} = 0 \tag{1.4}$$

$$\frac{dU}{d\theta} = k_r\,\theta - P_{cr}\,L\,\theta \tag{1.5}$$

$$\frac{dU}{d\theta} = \left[k_r - P_{cr}\;L\right]\theta \tag{1.6}$$

The governing equation is thus

$$\left[k_r - P_{cr}L\right]\theta = 0 \tag{1.7}$$

Set the determinant equal to zero.

$$\det\left[k_r - P_{cr}\;L\right] = 0 \tag{1.8}$$

Equation (1.8) can be solved directly for P_{cr}.

An alternative method is to apply equation (2), restated as equation (1.9), with an arbitrary load P. Note that this alternative method is often employed in finite element analysis of complex structures.

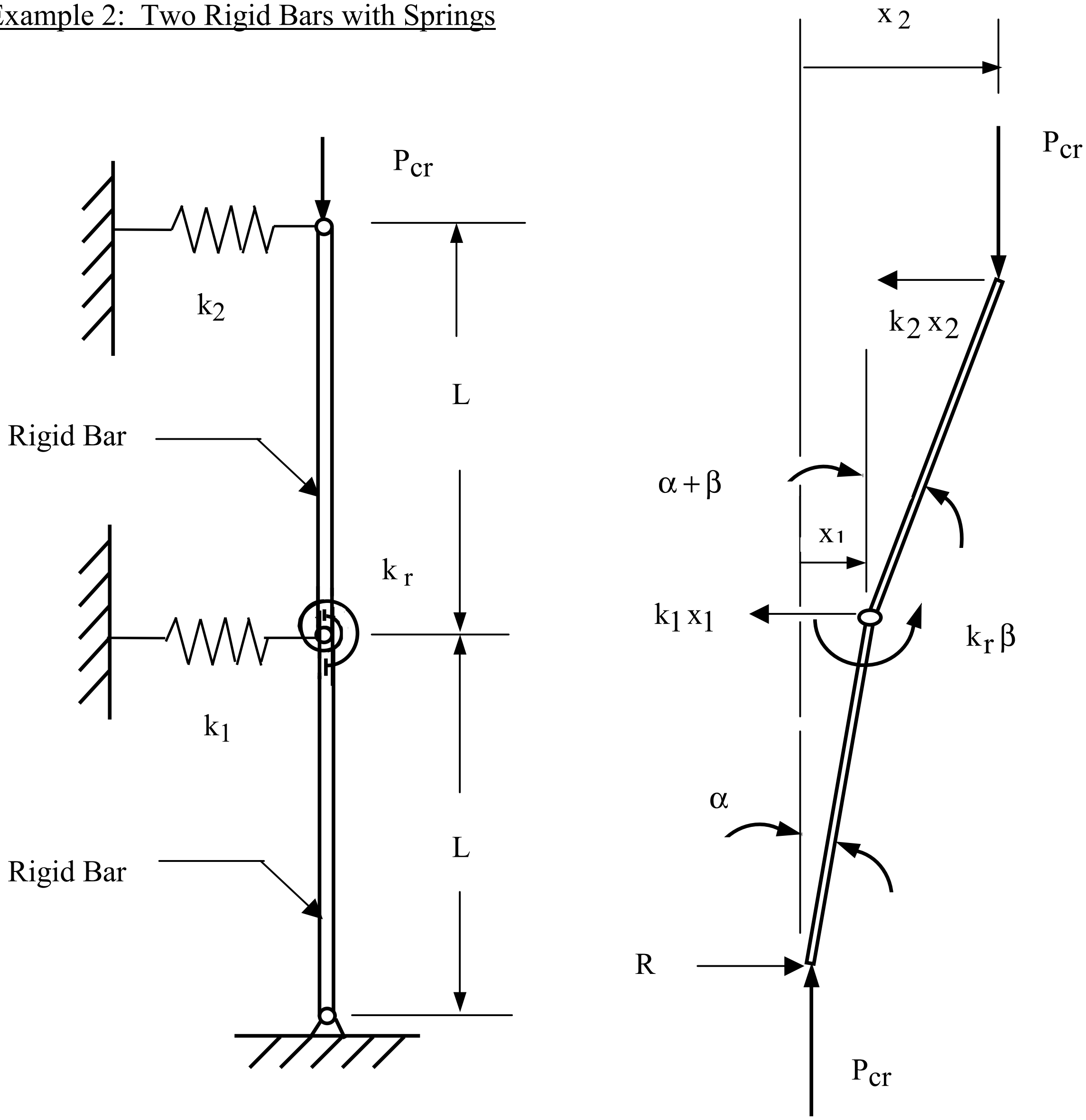
$$P_{cr} = \lambda_i P \tag{1.9}$$

$$\det\left[k_r - \lambda P L\right] = 0 \tag{1.10}$$

$$\lambda P = \frac{k_r}{L} \tag{1.11}$$

$$P_{cr} = \frac{k_r}{L} \tag{1.12}$$

Example 2: Two Rigid Bars with Springs



$$\alpha \approx \frac{x_1}{L} \tag{2.1}$$

$$\sin(\alpha + \beta) \approx \frac{x_2}{L} - \frac{x_1}{L} \tag{2.2}$$

For small angles,

$$(\alpha + \beta) \approx \frac{x_2}{L} - \frac{x_1}{L} \tag{2.3}$$

$$\beta \approx \frac{x_2}{L} - \frac{x_1}{L} - \alpha \tag{2.4}$$

$$\beta \approx \frac{x_2}{L} - \frac{x_1}{L} - \frac{x_1}{L} \tag{2.5}$$

$$\beta \approx \frac{x_2}{L} - \frac{2\,x_1}{L} \tag{2.6}$$

$$U = \frac{1}{2}k_1\,x_1{}^2 + \frac{1}{2}k_2\,x_2{}^2 + \frac{1}{2}k_r\,\beta^2 - P_{cr}\,(2 - \cos\alpha - \cos(\alpha+\beta))L \tag{2.7}$$

Note that for a small angle

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \tag{2.8}$$

$$U = \frac{1}{2}k_1\,x_1{}^2 + \frac{1}{2}k_2\,x_2{}^2 + \frac{1}{2}k_r\beta^2 - P_{cr}\left(2 - 1 + \frac{\alpha^2}{2} - 1 + \frac{(\alpha+\beta)^2}{2}\right)L \tag{2.9}$$

$$U = \frac{1}{2}k_1\,x_1{}^2 + \frac{1}{2}k_2\,x_2{}^2 + \frac{1}{2}k_r\beta^2 - P_{cr}\left(\frac{\alpha^2}{2} + \frac{(\alpha+\beta)^2}{2}\right)L \tag{2.10}$$

$$U = \frac{1}{2}k_{\,1}x_1{}^2 + \frac{1}{2}k_2\,x_2{}^2 + \frac{1}{2}k_r\left(\frac{x_2 - 2x_1}{L}\right)^2 - \frac{P_{cr}}{2L}\left(x_1{}^2 + (x_2 - x_1)^2\right) \tag{2.11}$$

$$U = \frac{1}{2}k_{\,1}x_1{}^2 + \frac{1}{2}k_2\,x_2{}^2 + \frac{1}{2}k_r\left(\frac{x_2 - 2x_1}{L}\right)^2 - \frac{P_{cr}}{2L}\left(x_1{}^2 + \left(x_2{}^2 - 2x_1x_2 + x_1{}^2\right)\right) \tag{2.12}$$

$$U = \frac{1}{2}k_1 x_1{}^2 + \frac{1}{2}k_2 x_2{}^2 + \frac{1}{2}k_r\left(\frac{x_2 - 2x_1}{L}\right)^2 - \frac{P_{cr}}{2L}\left(x_2{}^2 - 2x_1 x_2 + 2x_1{}^2\right) \tag{2.13}$$

$$\frac{\partial}{\partial x_1}U = k_1 x_1 - 2k_r\left(\frac{x_2 - 2x_1}{L^2}\right) - \frac{P_{cr}}{2L}(-2 + 4x_1) \tag{2.14}$$

$$\frac{\partial}{\partial x_1}U = k_1 x_1 + \left(\frac{-2k_r x_2 + 4k_r x_1}{L^2}\right) - \frac{P_{cr}}{2L}(-2x_2 + 4x_1) \tag{2.15}$$

$$\frac{\partial}{\partial x_1}U = \left(k_1 + \frac{4k_r}{L}\right)x_1 - \frac{2k_r x_2}{L^2} - \frac{P_{cr}}{L}(-x_2 + 2x_1) \tag{2.16}$$

$$\frac{\partial}{\partial x_2}U = +k_2 x_2 + k_r\left(\frac{x_2 - 2x_1}{L^2}\right) - \frac{P_{cr}}{2L}(2x_2 - 2x_1) \tag{2.17}$$

$$\frac{\partial}{\partial x_2}U = +k_2 x_2 + \frac{k_r x_2}{L^2} - \frac{2k_r}{L^2}x_1 - \frac{P_{cr}}{2L}(2x_2 - 2x_1) \tag{2.18}$$

$$\frac{\partial}{\partial x_2}U = \left[k_2 + \frac{k_r}{L^2}\right]x_2 - \frac{2k_r}{L^2}x_1 - \frac{P_{cr}}{L}(x_2 - x_1) \tag{2.19}$$

Apply the principle of stationary potential energy.

$$\frac{\partial U}{\partial x_1} = 0 \tag{2.20}$$

$$\frac{\partial U}{\partial x_2} = 0 \tag{2.21}$$

$$\left(k_1 + \frac{4k_r}{L}\right)x_1 - \frac{2k_r x_2}{L^2} - \frac{P_{cr}}{L}(-x_2 + 2x_1) = 0 \tag{2.22}$$

$$\left[k_2 + \frac{k_r}{L^2}\right]x_2 - \frac{2k_r}{L^2}x_1 - \frac{P_{cr}}{L}(x_2 - x_1) = 0 \tag{2.23}$$

$$\begin{bmatrix} k_1 + \frac{4k_r}{L^2} & -\frac{2k_r}{L^2} \\ -\frac{2k_r}{L^2} & k_2 + \frac{k_r}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{P_{cr}}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.24}$$

$$\begin{bmatrix} k_1 L + (4k_r / L) & -2k_r / L \\ -2k_r / L & k_2 L + (k_r / L) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - P_{cr}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.25}$$

$$\left\{\begin{bmatrix} k_1 L + (4k_r / L) & -2k_r / L \\ -2k_r / L & k_2 L + (k_r / L) \end{bmatrix} - P_{cr}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right\}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.26}$$

Set the determinant equal to zero.

$$\det\left\{\begin{bmatrix} k_1 L + (4k_r / L) & -2k_r / L \\ -2k_r / L & k_2 L + (k_r / L) \end{bmatrix} - P_{cr}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right\} = 0 \tag{2.27}$$

Equation (2) can be applied to equation (2.27) for an arbitrary load case P.

$$\det\left\{\begin{bmatrix} k_1 L + (4k_r / L) & -2k_r / L \\ -2k_r / L & k_2 L + (k_r / L) \end{bmatrix} - \lambda P\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right\} = 0 \tag{2.28}$$

Example 3: Elastic Column with Applied Discrete Load

The governing differential equation for a column is given by

$$\frac{d^4 y}{dx^4} + k^2 \frac{d^2 y}{dx^2} = 0 \tag{3.1}$$

where

$$k^2 = P / EI$$

The product EI is assumed to be constant.

Equation (3.1) is taken from Reference 1. It is a homogeneous, forth order, ordinary differential equation.

The weighted residual method is applied to equation (3.1). This method is suitable for boundary value problems. An alternative method would be the energy method.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (3.1) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of Y and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral is set equation to zero.

$$\int \phi(x) \left\{ \frac{d^4 y}{dx^4} + k^2 \frac{d^2 y}{dx^2} \right\} dx = 0 \tag{3.2}$$

The test function $\phi(x)$ can be regarded as a virtual displacement. The differential equation in the brackets represents an internal force. This term is also regarded as the residual. Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Define the domain over the limits from a to b. These limits represent the boundary points of the entire column.

$$\int_a^b \phi(x) \left\{ \frac{d^4 y}{dx^4} + k^2 \frac{d^2 y}{dx^2} \right\} dx = 0 \tag{3.3}$$

$$\int_a^b \phi(x) \left\{ \frac{d^4 y}{dx^4} \right\} dx + k^2 \int_a^b \phi(x) \left\{ \frac{d^2 y}{dx^2} \right\} dx = 0 \tag{3.4}$$

Integrate by parts.

$$\int_a^b \frac{d}{dx} \left\{ \phi(x) \frac{d^3}{dx^3} Y(x) \right\} dx - \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d^3}{dx^3} Y(x) \right\} dx$$
$$+ k^2 \int_a^b \frac{d}{dx} \left[\phi(x) \frac{d}{dx} Y(x) \right] dx - k^2 \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} dx = 0 \tag{3.5}$$

$$\left\{ \phi(x) \frac{d^3}{dx^3} Y(x) \right\} \Bigg|_a^b - \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d^3}{dx^3} Y(x) \right\} dx$$
$$+ k^2 \, \phi(x) \frac{d}{dx} Y(x) \Bigg|_a^b - k^2 \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} dx = 0 \tag{3.6}$$

Integrate by parts again.

$$\left\{ \phi(x) \frac{d^3}{dx^3} Y(x) \right\} \Bigg|_a^b - \int_a^b \frac{d}{dx} \left\{ \left[\frac{d}{dx} \phi(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx$$

$$+ \int_a^b \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$

$$+ k^2 \phi(x) \frac{d}{dx} Y(x) \Bigg|_a^b - k^2 \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} dx = 0$$

(3.7)

$$\left\{ \phi(x) \frac{d^3}{dx^3} Y(x) \right\} \Bigg|_a^b - \left\{ \left[\frac{d}{dx} \phi(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} \Bigg|_a^b$$

$$+ \int_a^b \left\{ \left[\frac{d^2}{dx^2} \phi(x) \right] \left[EI(x) \frac{d^2}{dx^2} Y(x) \right] \right\} dx$$

$$+ k^2 \phi(x) \frac{d}{dx} Y(x) \Bigg|_a^b - k^2 \int_a^b \left\{ \frac{d}{dx} \phi(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} dx = 0$$

(3.8)

Consider that the boundary conditions are either fixed or hinged. This assumption applies to equations (3.9) through (3.10).

Note that $Y(x) = 0$ at the boundaries.

$$\left\{ \phi(x) \frac{d^3}{dx^3} Y(x) \right\} \Bigg|_a^b = 0 \qquad (3.9)$$

And

$$k^2 \phi(x) \frac{d}{dx} Y(x) \Big|_a^b = 0 \tag{3.10}$$

Note that either $\frac{d}{dx} Y(x) = 0$ or $\frac{d^2}{dx^2} Y(x) = 0$ at the boundaries.

$$\left\{ \frac{d}{dx} \phi(x) \frac{d^2}{dx^2} Y(x) \right\} \Big|_a^b = 0 \tag{3.11}$$

A free boundary has the following condition, assuming that the end at x=L is free.

$$\left[\frac{d^3}{dx^3} Y(x) + k^2 \frac{d}{dx} Y(x) \right] \Big|_{x=L} = 0 \tag{3.12}$$

This boundary condition is taken from Reference 1, page 21.

An assumption is made for the free boundary condition that

$$\left[\frac{d^3}{dx^3} Y(x) \right] \Big|_{x=L} \approx 0 \tag{3.13}$$

$$\left[\frac{d}{dx} Y(x) \right] \Big|_{x=L} \approx 0 \tag{3.14}$$

Thus equations (3.10) and (3.11) are assumed to apply for the free boundary condition. On the other hand, the free boundary condition requires that

$$Y(L) \neq 0 \tag{3.15}$$

The approximations in equations (3.13) and (3.14) are rather tentative.

This tutorial will later show that the effect of the $\left.\left[\frac{d}{dx}Y(x)\right]\right|_{x=L}$ term is diminished as the element length decrease.

Nevertheless, a more rigorous justification is desired. The author continues to search for one.

Thus equation (3.8) becomes

$$\int_a^b \left\{ \left[\frac{d^2}{dx^2}\phi(x)\right]\left[\frac{d^2}{dx^2}Y(x)\right] \right\} dx \; - k^2\int_a^b \left\{\frac{d}{dx}\phi(x)\right\}\left\{ \frac{d}{dx}Y(x) \right\} dx \;=\; 0 \tag{3.16}$$

Now express the displacement function Y(x) in terms of nodal displacements y_{j-1} and y_j as well as the rotations θ_{j-1} and θ_j .

$$Y(x) = L_1\, y_{j-1} + L_2\, y_j \; + L_3\, h\theta_{j-1} + L_4\, h\theta_{j-1}\,, \qquad (j-1)h \;\le x \le jh \tag{3.17}$$

Note that h is the element length. In addition, each L coefficients is a function of x.

Now introduce a nondimensional natural coordinate ξ.

$$\xi = j - x/h \tag{3.18}$$

Note that h is the segment length.

The displacement function becomes.

$$Y(\xi) = L_1\, y_{j-1} + L_2\, h\theta_{j-1} + L_3\, y_j \; + L_4\, h\theta_{j-1}\,, \qquad 0 \;\le \xi \le 1 \tag{3.19}$$

The slope equation is

$$Y\,'(\xi) = L_1'\, y_{j-1} + L_2'\, h\theta_{j-1} + L_3'\, y_j \; + L_4'\, h\theta_{j-1}\,, \qquad 0 \;\le \xi \le 1 \tag{3.20}$$

The displacement function is represented terms of natural coordinates in Figure 3.1.

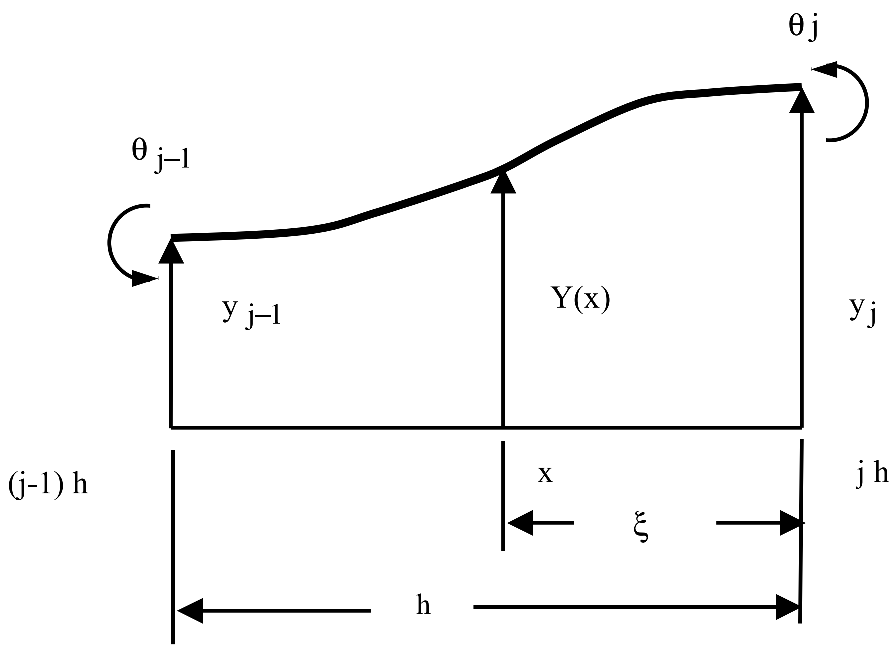


Figure 3.1

Represent each L coefficient in terms of a cubic polynomial.

$$L_i = c_{i1} + c_{i2}\xi + c_{i3}\xi^2 + c_{i4}\xi^3, \qquad i = 1, 2, 3, 4$$

(3.21)

$$\begin{aligned} Y(\xi) = & \left\{c_{11} + c_{12}\xi + c_{13}\xi^2 + c_{14}\xi^3\right\} y_{j-1} \\ & + \left\{c_{21} + c_{22}\xi + c_{23}\xi^2 + c_{24}\xi^3\right\} h\,\theta_{j-1} \\ & + \left\{c_{31} + c_{32}\xi + c_{33}\xi^2 + c_{34}\xi^3\right\} y_j \\ & + \left\{c_{41} + c_{42}\xi + c_{43}\xi^2 + c_{44}\xi^3\right\} h\,\theta_j, \qquad 0 \le \xi \le 1 \end{aligned}$$

(3.22)

A complete derivation of the coefficients is given in Reference 3. The displacement equation becomes

$$Y(\xi) = + \left\{3\,\xi^2 - 2\xi^3\right\} y_{j-1} + \left\{\,\xi^2 - \xi^3\right\} h\,\theta_{j-1} + \left\{1 - 3\,\xi^2 + 2\xi^3\right\} y_j + \left\{-\xi + 2\xi^2 - \xi^3\right\} h\,\theta_j \;, \qquad 0 \le \xi \le 1 \tag{3.23}$$

Recall

$$\xi = j - x/h \tag{3.24}$$

Thus

$$d\xi = -dx/h \tag{3.25}$$

$$-h\,d\xi = dx \tag{3.26}$$

$$\frac{d\xi}{dx} = -1/h \tag{3.27}$$

Note

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} \tag{3.28}$$

$$Y(x) = + \left\{3\,\xi^2 - 2\xi^3\right\} y_{j-1} + \left\{\,\xi^2 - \xi^3\right\} h\,\theta_{j-1} + \left\{1 - 3\,\xi^2 + 2\xi^3\right\} y_j + \left\{-\xi + 2\xi^2 - \xi^3\right\} h\,\theta_j \;,$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0 \le \xi \le 1 \tag{3.29}$$

$$\frac{d}{dx}Y(x) = \{-1/h\}\Big\{ \Big[\ 6\,\xi - 6\xi^2 \Big]\, y_{j-1} + \Big[2\xi - 3\xi^2 \Big]\, h\,\theta_{j-1} + \Big[-6\ \xi + 6\xi^2 \Big]\, y_j\ + \Big[-1 + 4\xi - 3\xi^2 \Big]\, h\,\theta_j \Big\},$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0\ \le \xi \le 1 \tag{3.30}$$

$$\frac{d^2}{dx^2}Y(x) = \Big\{1/h^2\Big\}\Big\{[\ 6 - 12\xi]\, y_{j-1} + [2 - 6\xi]\, h\,\theta_{j-1} + [\ -6\ + 12\xi]\, y_j\ + [\ 4 - 6\ \xi]\, h\,\theta_j\ \Big\},$$

$$(j-1)h \le x \le jh, \quad \xi = j - x/h, \quad 0\ \le \xi \le 1 \tag{3.31}$$

Now Let

$$Y(x) = \underline{L}^T\ \ \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{3.32}$$

where

$$L_1 = 3\,\xi^2 - 2\xi^3 \tag{3.33}$$

$$L_2 = \xi^2 - \xi^3 \tag{3.34}$$

$$L_3 = 1 - 3\ \xi^2 + 2\xi^3 \tag{3.35}$$

$$L_4 = -\xi + 2\xi^2 - \xi^3 \tag{3.36}$$

$$\bar{a} = [\ \ y_{j-1} \quad h\,\theta_{j-1} \quad y_j \quad h\,\theta_j\ \ ]^T \tag{3.37}$$

The derivative terms are

$$\frac{d}{dx}Y(x) = \left(\frac{-1}{h}\right)\ \underline{L'}^{\,T}\ \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{3.38}$$

$$\frac{d^2}{dx^2}Y(x) = \left(\frac{1}{h^2}\right) \underline{L}''^{\,T} \quad \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{3.39}$$

Note that primes indicate derivatives with respect to ξ.

In summary.

$$\underline{L} = \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \tag{3.40}$$

$$\underline{L}' = \begin{bmatrix} 6\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix} \tag{3.41}$$

$$\underline{L}'' = \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6 + 12\xi \\ 4 - 6\xi \end{bmatrix} \tag{3.42}$$

Recall

$$\int_a^b \left\{ \left[\frac{d^2}{dx^2}\phi(x) \right] \left[\frac{d^2}{dx^2}Y(x) \right] \right\} dx \; - k^2 \int_a^b \left\{ \frac{d}{dx}\phi(x) \right\} \left\{ \frac{d}{dx}Y(x) \right\} dx \; = \; 0 \tag{3.43}$$

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = Y(x) \tag{3.44}$$

Thus

$$\int_a^b \left\{ \left[\frac{d^2}{dx^2} Y(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx \; - k^2 \int_a^b \left\{ \frac{d}{dx} Y(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} dx \; = \; 0 \tag{3.45}$$

$$\int_a^b \left\{ \left[\frac{d^2}{dx^2} Y(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} dx \; - k^2 \int_a^b \left\{ \frac{d}{dx} Y(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} dx \; = \; 0 \tag{3.46}$$

Change the integration variable using equation (3.46). Also, apply the integration limits.

$$h \int_0^1 \left\{ \left[\frac{d^2}{dx^2} Y(x) \right] \left[\frac{d^2}{dx^2} Y(x) \right] \right\} d\xi - h k^2 \int_0^1 \left\{ \frac{d}{dx} Y(x) \right\} \left\{ \frac{d}{dx} Y(x) \right\} d\xi = 0 \tag{3.47}$$

The following transformation is taken from Reference 3.

Let

$$Y(x) = \underline{L}^T \; \bar{a} , \qquad (j-1)h \le x \le jh, \quad \xi = j - x / h \tag{3.48}$$

where

$$L_1 = 3\,\xi^2 - 2\xi^3 \tag{3.49}$$

$$L_2 = \xi^2 - \xi^3 \tag{3.50}$$

$$L_3 = 1 - 3\,\xi^2 + 2\xi^3 \tag{3.51}$$

$$L_4 = -\xi + 2\xi^2 - \xi^3 \tag{3.52}$$

$$\bar{a} = \begin{bmatrix} y_{j-1} & h\theta_{j-1} & y_j & h\theta_j \end{bmatrix}^T \tag{3.53}$$

The derivative terms are

$$\frac{d}{dx} Y(x) = \left(\frac{-1}{h} \right) \underline{L}'^{\,T} \, \bar{a} , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{3.54}$$

$$\frac{d^2}{dx^2} Y(x) = \left(\frac{1}{h^2} \right) \underline{L}''^{\,T} \, \bar{a} , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{3.55}$$

Note that primes indicate derivatives with respect to ξ.

In summary.

$$\underline{L} = \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \tag{3.56}$$

$$\underline{L}' = \begin{bmatrix} 6\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix} \tag{3.57}$$

$$\underline{L}'' = \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6 + 12\xi \\ 4 - 6\xi \end{bmatrix} \tag{3.58}$$

Recall

$$h \int_0^1 \left\{ \left[\left(\frac{1}{h^2}\right) \underline{L''}^T \bar{a} \right] \left[\left(\frac{1}{h^2}\right) \underline{L''}^T \bar{a} \right] \right\} d\xi$$

$$-h k^2 \int_0^1 \left[\left(\frac{1}{h}\right) \underline{L'}^T \bar{a} \right] \left[\left(\frac{1}{h}\right) \underline{L'}^T \bar{a} \right] d\xi = 0$$

(3.59)

$$\left(\frac{1}{h^3}\right) \int_0^1 \left\{ \left[\underline{L''}^T \bar{a} \right] \left[\underline{L''}^T \bar{a} \right] \right\} d\xi$$

$$-\left(\frac{k^2}{h}\right) \int_0^1 \left[\underline{L'}^T \bar{a} \right] \left[\underline{L'}^T \bar{a} \right] d\xi = 0$$

(3.60)

$$\left(\frac{1}{h^3}\right) \int_0^1 \left\{ \left[\bar{a}^T \underline{L''} \right] \left[\underline{L''}^T \bar{a} \right] \right\} d\xi$$

$$-\left(\frac{k^2}{h}\right) \int_0^1 \left[\bar{a}^T \underline{L'} \right] \left[\underline{L'}^T \bar{a} \right] d\xi = 0$$

(3.61)

$$\left(\frac{1}{h^3}\right) \int_0^1 \left\{ \bar{a}^T \underline{L''}\, \underline{L''}^T \bar{a} \right\} d\xi - \left(\frac{k^2}{h}\right) \int_0^1 \left\{ \bar{a}^T \underline{L'}\, \underline{L'}^T \bar{a} \right\} d\xi = 0$$

(3.62)

$$\bar{a}^T\left\{\left(\frac{1}{h^3}\right)\int_0^1\left\{\underline{L}''\ \underline{L}''^T\right\}d\xi-\left(\frac{k^2}{h}\right)\int_0^1\left\{\underline{L}'\ \underline{L}'^T\right\}d\xi\right\}\bar{a}=0 \qquad (3.63)$$

$$\left(\frac{1}{h^3}\right)\int_0^1\left\{\underline{L}''\ \underline{L}''^T\right\}d\xi-\left(\frac{k^2}{h}\right)\int_0^1\left\{\underline{L}'\ \underline{L}'^T\right\}d\xi=0 \qquad (3.64)$$

For a system of n elements,

$$K_j-k^2\,G_j=0\,,\qquad j=1,2,\ldots,n \qquad (3.65)$$

where

$$K_j=\left(\frac{1}{h^3}\right)\int_0^1\left\{\underline{L}''\ \underline{L}''^T\right\}d\xi \qquad (3.66)$$

$$G_j=\left(\frac{1}{h}\right)\int_0^1\left\{\underline{L}'\ \underline{L}'^T\right\}d\xi \qquad (3.67)$$

This linear stiffness term is derived in Reference 3.

$$K_j=$$

$$\left(\frac{1}{h^3}\right)\int_0^1\begin{bmatrix}36-144\xi+144\xi^2 & 12-60\xi+72\xi^2 & -36+144\xi-144\xi^2 & 24-84\xi+72\xi^2\\ & 4-24\xi+36\xi^2 & -12+60\xi-72\xi^2 & 8-36\xi+36\xi^2\\ & & 36-144\xi+144\xi^2 & -24+84\xi-72\xi^2\\ & & & 16-48\xi+36\xi^2\end{bmatrix}d\xi$$

(3.68)

Note that only the upper triangular components are shown due to symmetry.

$$K_j = \left(\frac{1}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix}$$

(3.69)

The global differential matrix is calculated as follows.

$$\underline{L}'\,\underline{L}'^T = \begin{bmatrix} 6\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\ \xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix}\begin{bmatrix} 6\xi - 6\xi^2 & 2\xi - 3\xi^2 & -6\ \xi + 6\xi^2 & -1 + 4\xi - 3\xi^2 \end{bmatrix}$$

(3.70)

$$\underline{L}'\,\underline{L}'^T = \begin{bmatrix} a11 & a12 & a13 & a14 \\ & a22 & a23 & a24 \\ & & a33 & a34 \\ & & & a44 \end{bmatrix}$$

(3.71)

$$a11 = \left(6\xi - 6\xi^2\right)\left(6\xi - 6\xi^2\right) \tag{3.72}$$

$$a11 = 36\xi^2 - 72\xi^3 + 36\xi^4 \tag{3.73}$$

$$a12 = \left(6\xi - 6\xi^2\right)\left(2\xi - 3\xi^2\right) \tag{3.74}$$

$$a12 = 12\xi^2 - 30\xi^3 + 18\xi^4 \tag{3.75}$$

$$a13 = \left(6\xi - 6\xi^2\right)\left(-6\xi + 6\xi^2\right) \tag{3.76}$$

$$a13 = -36\xi^2 + 72\xi^3 - 36\xi^4 \tag{3.77}$$

$$a14 = \left(6\xi - 6\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right) \tag{3.78}$$

$$a14 = -6\xi + 6\xi^2 + 24\xi^2 - 24\xi^3 - 18\xi^3 + 18\xi^4 \tag{3.79}$$

$$a14 = -6\xi + 30\xi^2 - 42\xi^3 + 18\xi^4 \tag{3.80}$$

$$a22 = \left(2\xi - 3\xi^2\right)\left(2\xi - 3\xi^2\right) \tag{3.81}$$

$$a22 = 4\xi^2 - 12\xi^3 + 9\xi^4 \tag{3.82}$$

$$a23 = \left(2\xi - 3\xi^2\right)\left(-6\xi + 6\xi^2\right) \tag{3.83}$$

$$a23 = -12\xi^2 + 30\xi^3 - 18\xi^4 \tag{3.84}$$

$$a24 = \left(2\xi - 3\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right) \tag{3.85}$$

$$a24 = -2\xi + 3\xi^2 + 8\xi^2 - 12\xi^3 - 6\xi^3 + 9\xi^4 \tag{3.86}$$

$$a24 = -2\xi + 11\xi^2 - 18\xi^3 + 9\xi^4 \tag{3.87}$$

$$a33 = \left(-6\xi + 6\xi^2\right)\left(-6\xi + 6\xi^2\right) \tag{3.88}$$

$$a33 = 36\xi^2 - 72\xi^3 + 36\xi^4 \tag{3.89}$$

$$a34 = \left(-6\xi + 6\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right) \tag{3.90}$$

$$a34 = 6\xi - 6\xi^2 - 24\xi^2 + 24\xi^3 + 18\xi^3 - 18\xi^4 \tag{3.91}$$

$$a34 = 6\xi - 30\xi^2 + 42\xi^3 - 18\xi^4 \quad (3.92)$$

$$a44 = \left(-1 + 4\xi - 3\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right) \quad (3.93)$$

$$a44 = 1 - 4\xi + 3\xi^2 - 4\xi + 16\xi^2 - 12\xi^3 + 3\xi^2 - 12\xi^3 + 9\xi^4 \quad (3.94)$$

$$a44 = 1 - 8\xi + 22\xi^2 - 24\xi^3 + 9\xi^4 \quad (3.95)$$

Recall

$$G_j = \left(\frac{1}{h}\right)\int_0^1 \left\{ \underline{L}' \underline{L}'^T \right\} d\xi \quad (3.96)$$

$$\int_0^1 \{ a11 \} \, d\xi = \int_0^1 \left\{ 36\xi^2 - 72\xi^3 + 36\xi^4 \right\} d\xi \quad (3.97)$$

$$\int_0^1 \{ a11 \} \, d\xi = \left\{ 12\xi^3 - 18\xi^4 + (36/5)\xi^5 \right\}\Big|_0^1 \quad (3.98)$$

$$\int_0^1 \{ a11 \} \, d\xi = 12 - 18 + (36/5) \quad (3.99)$$

$$\int_0^1 \{ a11 \} \, d\xi = \frac{6}{5} \quad (3.100)$$

$$\int_0^1 \{ a12 \} \, d\xi = \int_0^1 \left\{ 12\xi^2 - 30\xi^3 + 18\xi^4 \right\} d\xi \quad (3.101)$$

$$\int_0^1 \{ a12 \} \, d\xi = 4\xi^3 - (15/2)\xi^4 + (18/5)\xi^5 \Big|_0^1 \quad (3.102)$$

$$\int_0^1 \{ \text{ a12 } \} \, d\xi = \quad 4-(15/2)+(18/5) \tag{3.103}$$

$$\int_0^1 \{ \text{ a12 } \} \, d\xi = \frac{1}{10} \tag{3.104}$$

$$\int_0^1 \{ \text{ a13 } \} \, d\xi = \int_0^1 \left\{ -36\xi^2 + 72\xi^3 - 36\xi^4 \right\} d\xi \tag{3.105}$$

$$\int_0^1 \{ \text{ a13 } \} \, d\xi = \quad -12\xi^3 + 18\xi^4 - (36/5)\xi^5 \Big|_0^1 \tag{3.106}$$

$$\int_0^1 \{ \text{ a13 } \} \, d\xi = -12+18-(36/5) \tag{3.107}$$

$$\int_0^1 \{ \text{ a13 } \} \, d\xi = -\frac{6}{5} \tag{3.108}$$

$$\int_0^1 \{a14\} d\xi = \int_0^1 \left\{ -6\xi + 30\xi^2 - 42\xi^3 + 18\xi^4 \right\} d\xi \tag{3.109}$$

$$\int_0^1 \{a14\} d\xi = \left\{ -3\xi^2 + 10\xi^3 - \frac{21}{2}\xi^4 + \frac{18}{5}\xi^5 \right\} \Bigg|_0^1 \tag{3.110}$$

$$\int_0^1 \{a14\} d\xi = \left\{ -3 + 10 - \frac{21}{2} + \frac{18}{5} \right\} \tag{3.111}$$

$$\int_0^1 \{a14\} d\xi = \frac{1}{10} \tag{3.112}$$

$$\int_0^1 \{ \text{ a22 } \} \, d\xi = \int_0^1 \left\{ 4\xi^2 - 12\xi^3 + 9\xi^4 \right\} d\xi \tag{3.113}$$

$$\int_0^1 \{\ a22\ \}\ d\xi = \left\{(4/3)\xi^3 - 3\xi^4 + (9/5)\xi^5\right\}\Big|_0^1 \tag{3.114}$$

$$\int_0^1 \{\ a22\ \}\ d\xi = (4/3) - 3 + (9/5) \tag{3.115}$$

$$\int_0^1 \{\ a22\ \}\ d\xi = \frac{2}{15} \tag{3.116}$$

$$\int_0^1 \{\ a23\ \}\ d\xi = \int_0^1 \left\{-12\xi^2 + 30\xi^3 - 18\xi^4\right\} d\xi \tag{3.117}$$

$$\int_0^1 \{\ a23\ \}\ d\xi = \left\{-4\xi^3 + (15/2)\xi^4 - (18/5)\xi^5\right\}\Big|_0^1 \tag{3.118}$$

$$\int_0^1 \{\ a23\ \}\ d\xi = -4 + (15/2) - (18/5) \tag{3.119}$$

$$\int_0^1 \{\ a23\ \}\ d\xi = -\frac{1}{10} \tag{3.120}$$

$$\int_0^1 \{\ a24\ \}\ d\xi = \int_0^1 \left\{-2\xi + 11\xi^2 - 18\xi^3 + 9\xi^4\right\} d\xi \tag{3.121}$$

$$\int_0^1 \{\ a24\ \}\ d\xi = \left\{-\xi^2 + (11/3)\xi^3 - (9/2)\xi^4 + (9/5)\xi^5\right\}\Big|_0^1 \tag{3.122}$$

$$\int_0^1 \{\ a24\ \}\ d\xi = \{-1 + (11/3) - (9/2) + (9/5)\} \tag{3.123}$$

$$\int_0^1 \{\ a24\ \}\ d\xi = -1/30 \tag{3.124}$$

$$\int_0^1 \{ \text{ a33 } \} \, d\xi = \int_0^1 \left\{ 36\xi^2 - 72\xi^3 + 36\xi^4 \right\} d\xi \tag{3.125}$$

$$\int_0^1 \{ \text{ a33 } \} \, d\xi = \left\{ 12\xi^3 - 18\xi^4 + (36/5)\xi^5 \right\} \Big|_0^1 \tag{3.126}$$

$$\int_0^1 \{ \text{ a33 } \} \, d\xi = 12 - 18 + (36/5) \tag{3.127}$$

$$\int_0^1 \{ \text{ a33 } \} \, d\xi = 6/5 \tag{3.128}$$

$$\int_0^1 \{ \text{ a34 } \} \, d\xi = \int_0^1 \left\{ 6\xi - 30\xi^2 + 42\xi^3 - 18\xi^4 \right\} d\xi \tag{3.129}$$

$$\int_0^1 \{ \text{ a34 } \} \, d\xi = \left\{ 3\xi^2 - 10\xi^3 + (21/2)\xi^4 - (18/5)\xi^5 \right\} \Big|_0^1 \tag{3.130}$$

$$\int_0^1 \{ \text{ a34 } \} \, d\xi = \{ 3 - 10 + (21/2) - (18/5) \} \tag{3.131}$$

$$\int_0^1 \{ \text{ a34 } \} \, d\xi = -1/10 \tag{3.132}$$

$$\int_0^1 \{ \text{ a44 } \} \, d\xi = \int_0^1 \left\{ 1 - 8\xi + 22\xi^2 - 24\xi^3 + 9\xi^4 \right\} d\xi \tag{3.133}$$

$$\int_0^1 \{ \text{ a44 } \} \, d\xi = \left\{ \xi - 4\xi^2 + (22/3)\xi^3 - 6\xi^4 + (9/5)\xi^5 \right\} \Big|_0^1 \tag{3.134}$$

$$\int_0^1 \{ \, a44 \, \} \, d\xi = \{ \, 1 - 4 + (22/3) - 6 + (9/5) \} \tag{3.135}$$

$$\int_0^1 \{ \, a44 \, \} \, d\xi = 2/15 \tag{3.136}$$

$$G_j = \left(\frac{1}{h}\right) \begin{bmatrix} \frac{6}{5} & \frac{1}{10} & \frac{-6}{5} & \frac{1}{10} \\ & \frac{2}{15} & \frac{-1}{10} & \frac{-1}{30} \\ & & \frac{6}{5} & \frac{-1}{10} \\ & & & \frac{2}{15} \end{bmatrix} \tag{3.137}$$

$$G_j = \left(\frac{1}{30h}\right) \begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix} \tag{3.138}$$

The linear stiffness matrix in equation (3.69) and the global differential matrix in equation (3.138) can be assembled to form global stiffness matrices for the generalized eigenvalue problem in equation (3.65).

As an example, consider the fixed-free column in Figure 3.2, with length L. Divide the column into eight elements of equal length. Assemble the global linear stiffness and global differential matrices. Then solve for the first eigenvalue using a numerical method such as the Jacobi method.

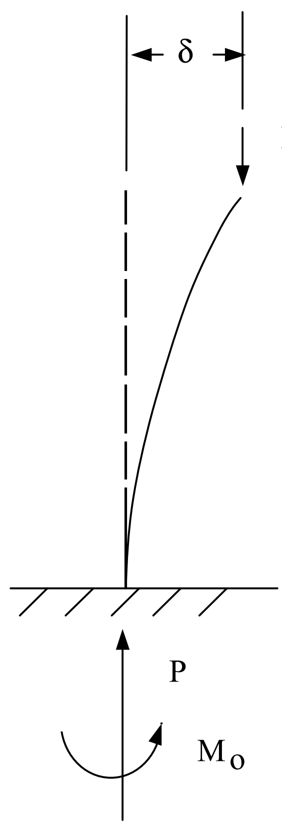


Figure 3.2

The mode shape corresponding to the first eigenvalue is shown in Figure 3.3.

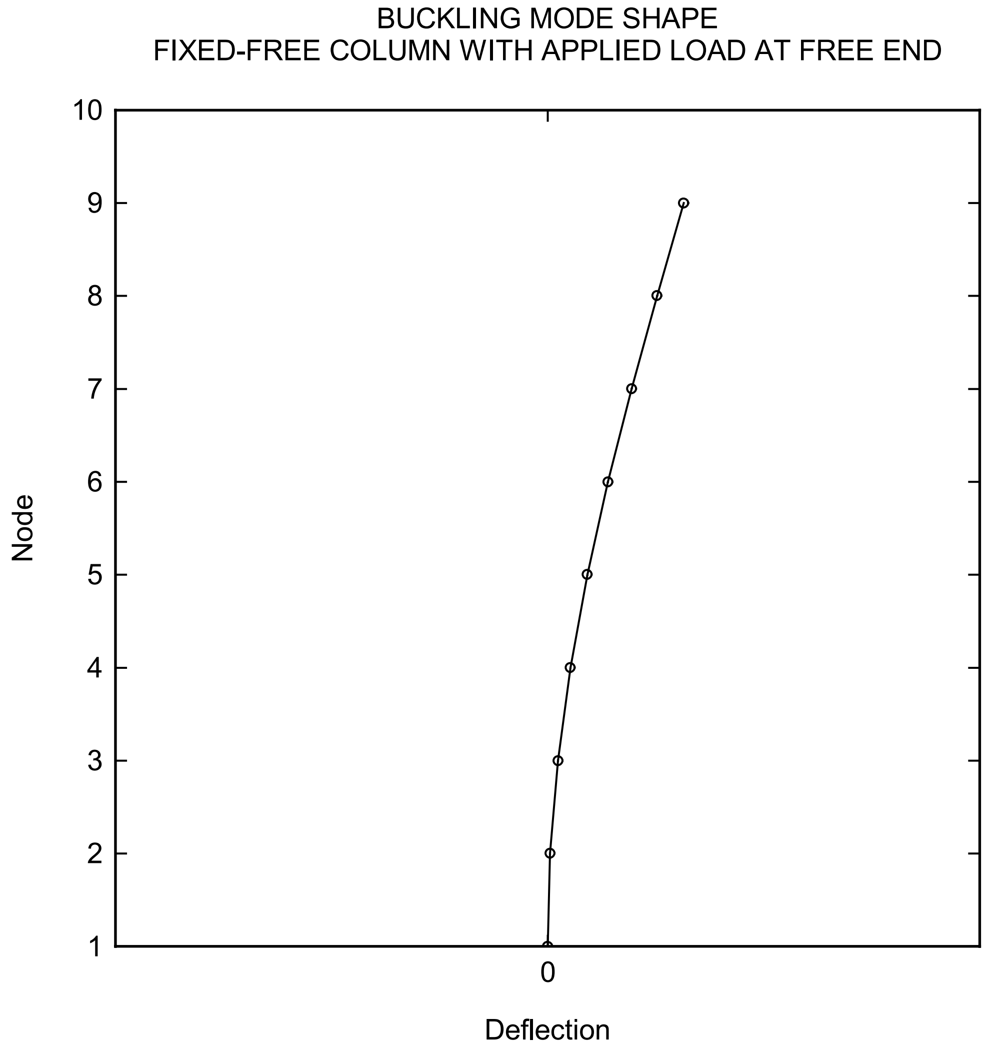


Figure 3.3

The first eigenvalue is

$$\frac{P_{cr}\ h^2}{EI} = 0.038543 \tag{3.139}$$

$$P_{cr} = 0.038543 \frac{EI}{h^2} \tag{3.140}$$

$$h = L\ /\ 8 \tag{3.141}$$

$$P_{cr} = 0.038543 \frac{EI}{(L/8)^2} \tag{3.142}$$

$$P_{cr} \;=\; 2.467\;\frac{EI}{L^2} \qquad \text{(via finite element method)} \qquad (3.143)$$

The theoretical value from References 1 and 2 is

$$P_{cr} = \frac{\pi^2\, EI}{4l^2} \;\approx\; 2.467\;\frac{EI}{L^2} \qquad \text{(theory)} \qquad (3.144)$$

Thus, the finite element method yields the exact answer to three decimal places.

Lower End Fixed, Upper End Free Column with Distributed Body Weight Load

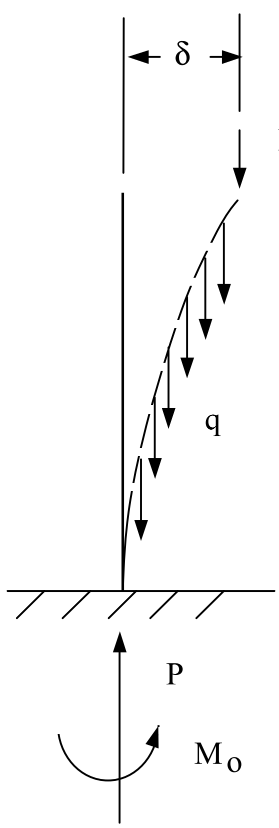


Figure 4.1

Model the column in Figure 4.1 with eight elements and nine nodes.

Let

$$P = q\,L \tag{4.1}$$

Model the distributed body load as a series of discrete nodal forces as shown in Figure 4.2.

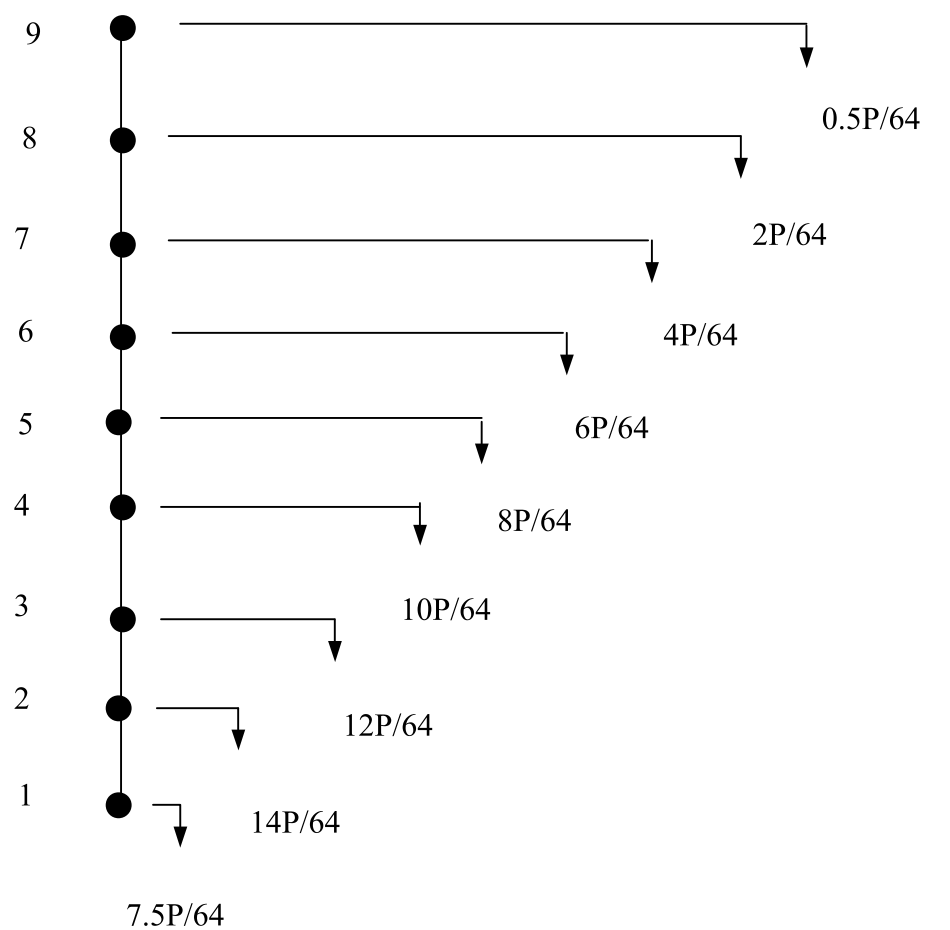


Figure 4-2.

Thus, each element will have a unique different stiffness matrix G_j.

The mode shape corresponding to the first eigenvalue is shown in Figure 4.3.

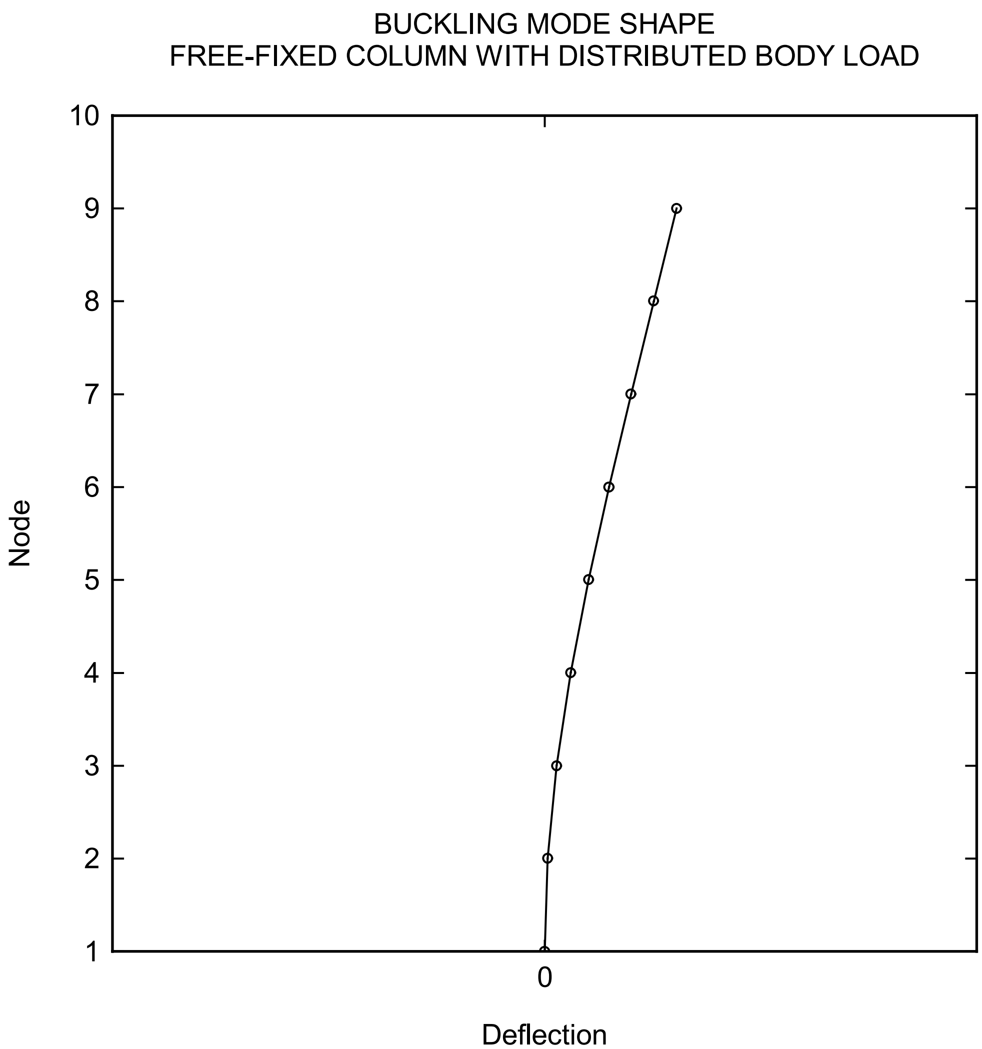


Figure 4.3

The first eigenvalue is

$$\frac{P_{cr}\ h^2}{EI} = 0.94806 \tag{4.2}$$

$$P_{cr} = 0.94806 \frac{EI}{h^2} \tag{4.3}$$

$$h = L / 8 \tag{4.4}$$

$$P_{cr} = 0.94806 \frac{EI}{(L/8)^2} \tag{4.5}$$

$$P_{cr} = 7.584 \frac{EI}{L^2} \tag{4.6}$$

$$(qL)_{cr} = 7.584 \frac{EI}{L^2} \quad \text{(finite element value)} \tag{4.7}$$

The theoretical value from Reference 1 is

$$(qL)_{cr} = \frac{7.837\ EI}{L^2} \quad \text{(theory)} \tag{4.8}$$

The finite element value is thus 3.2% lower than the theoretical value.

References

1. Timoshenko and Gere, Theory of Elastic Stability, International Student Edition, 2nd Edition, McGraw-Hill, New Delhi, 1963.
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3. Irvine, Transverse Vibration of A Beam Via The Finite Element Method, Vibration Data Publications, 2000.