STEADY-STATE VIBRATION RESPONSE OF A
CANTILEVER BEAM SUBJECTED TO BASE EXCITATION
Revision D

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Normal Modes

Consider the cantilever beam in Figure 1.

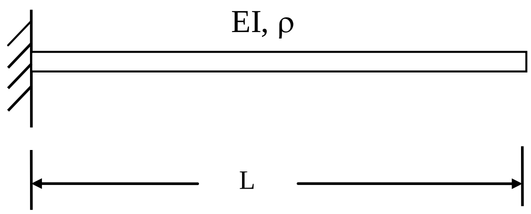

Figure 1.

The governing differential equation for the displacement y(x,t) is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)

Note that this equation neglects shear deformation and rotary inertia.

Separate the dependent variable.

$$y(x,t) = Y(x)T(t) \tag{2}$$

$$-EI\frac{\partial^4[Y(x)T(t)]}{\partial x^4} = \rho\frac{\partial^2[Y(x)T(t)]}{\partial t^2} \tag{3}$$

$$-EI\ T(t)\left\{\frac{d^4}{dx^4}Y(x)\right\} = \rho\ Y(x)\left\{\frac{d^2}{dt^2}T(t)\right\} \tag{4}$$

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = \frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} \tag{5}$$

Let c be a constant

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = \frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} = -c^2 \tag{6}$$

Separate the time variable.

$$\frac{\left\{\frac{d^2}{dt^2}T(t)\right\}}{T(t)} = -c^2 \tag{7}$$

$$\frac{d^2}{dt^2}T(t) + c^2\ T(t) = 0 \tag{8}$$

Separate the spatial variable.

$$\left\{\frac{-EI}{\rho}\right\}\frac{\left\{\frac{d^4}{dx^4}Y(x)\right\}}{Y(x)} = -c^2 \tag{9}$$

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{10}$$

A solution for equation (10) is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{11}$$

$$\frac{dY(x)}{dx} = a_1\beta \cosh(\beta x) + a_2\beta \sinh(\beta x) + a_3\beta \cos(\beta x) - a_4\beta \sin(\beta x) \tag{12}$$

$$\frac{d^2\,Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \tag{13}$$

$$\frac{d^3\,Y(x)}{dx^3} = a_1\beta^3 \cosh(\beta x) + a_2\beta^3 \sinh(\beta x) - a_3\beta^3 \cos(\beta x) + a_4\beta^3 \sin(\beta x) \tag{14}$$

$$\frac{d^4\,Y(x)}{dx^4} = a_1\beta^4 \sinh(\lambda x) + a_2\beta^4 \cosh(\beta x) + a_3\beta^4 \sin(\beta x) + a_4\beta^4 \cos(\beta x) \tag{15}$$

Substitute (15) and (11) into (10).

$$\left\{a_1\beta^4 \sinh(\beta x) + a_2\beta^4 \cosh(\beta x) + a_3\beta^4 \sin(\beta x) + a_4\beta^4 \cos(\beta x)\right\}$$
$$- c^2\left\{\frac{\rho}{EI}\right\}\left\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\right\} = 0$$

(16)

$$\beta^4 \left\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\right\}$$
$$- c^2 \left\{\frac{\rho}{EI}\right\} \left\{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\right\} = 0 \tag{17}$$

The equation is satisfied if

$$\beta^4 = c^2 \left\{\frac{\rho}{EI}\right\} \tag{18}$$

$$\beta = \left\{c^2 \frac{\rho}{EI}\right\}^{1/4} \tag{19}$$

The boundary conditions at the fixed end x = 0 are

$$Y(0) = 0 \qquad \text{(zero displacement)} \tag{20}$$

$$\left.\frac{dY}{dx}\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{21}$$

The boundary conditions at the free end x = L are

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{22}$$

$$\left.\frac{d^3Y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{23}$$

Apply equation (20) to (11).

$$a_2 + a_4 = 0 \tag{24}$$

$$a_4 = -a_2 \tag{25}$$

Apply equation (21) to (12).

$$a_1 + a_3 = 0 \tag{26}$$

$$a_3 = -a_1 \tag{27}$$

Apply equation (22) to (13).

$$a_1 \sinh(\beta L) + a_2 \cosh(\beta L) - a_3 \sin(\beta L) - a_4 \cos(\beta L) = 0 \tag{28}$$

Apply equation (23) to (14).

$$a_1 \cosh(\beta L) + a_2 \sinh(\beta L) - a_3 \cos(\beta L) + a_4 \sin(\beta L) = 0 \tag{29}$$

Apply (25) and (27) to (28).

$$a_1 \sinh(\beta L) + a_2 \cosh(\beta L) + a_1 \sin(\beta L) + a_2 \cos(\beta L) = 0 \tag{30}$$

$$a_1 \{\sin(\beta L) + \sinh(\beta L)\} + a_2 \{\cos(\beta L) + \cosh(\beta L)\} = 0 \tag{31}$$

Apply (25) and (27) to (29).

$$a_1 \cosh(\beta L) + a_2 \sinh(\beta L) + a_1 \cos(\beta L) - a_2 \sin(\beta L) = 0 \tag{32}$$

$$a_1 \{\cos(\beta L) + \cosh(\beta L)\} + a_2 \{-\sin(\beta L) + \sinh(\beta L)\} = 0 \tag{33}$$

Form (31) and (33) into a matrix format.

$$\begin{bmatrix} \sin(\beta L) + \sinh(\beta L) & \cos(\beta L) + \cosh(\beta L) \\ \cos(\beta L) + \cosh(\beta L) & -\sin(\beta L) + \sinh(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{34}$$

By inspection, equation (34) can only be satisfied if $a_1 = 0$ and $a_2 = 0$. Set the determinant to zero in order to obtain a nontrivial solution.

$$\{-\sin^2(\beta L) + \sinh^2(\beta L)\} - \{\cos(\beta L) + \cosh(\beta L)\}^2 = 0 \tag{35}$$

$$\left\{-\sin^2(\beta L)+\sinh^2(\beta L)\right\}-\left\{\cos^2(\beta L)+2\cos(\beta L)\cosh(\beta L)+\cosh^2(\beta L)\right\}=0 \tag{36}$$

$$-\sin^2(\beta L)+\sinh^2(\beta L)-\cos^2(\beta L)-2\cos(\beta L)\cosh(\beta L)-\cosh^2(\beta L)=0 \tag{37}$$

$$-2-2\cos(\beta L)\cosh(\beta L)=0 \tag{38}$$

$$1+\cos(\beta L)\cosh(\beta L)=0 \tag{39}$$

$$\cos(\beta L)\cosh(\beta L)=-1 \tag{40}$$

There are multiple roots which satisfy equation (40). Thus, a subscript should be added as shown in equation (41).

$$\cos(\beta_n L)\cosh(\beta_n L)=-1 \tag{41}$$

The subscript is an integer index. The roots can be determined through a combination of graphing and numerical methods. The Newton-Raphson method is an example of an appropriate numerical method. The roots of equation (41) are summarized in Table 1, as taken from Reference 1.

Table 1. Roots	
Index	$\beta_n L$
n = 1	1.87510
n = 2	4.69409
n = 3	7.85476
n = 4	10.99554
$n \geq 5$	$(2n-1)\pi/2$

Note: the root value formula for $n \geq 5$ is approximate.

Rearrange equation (19) as follows

$$c^2 = \beta_n{}^4 \left[\frac{EI}{\rho}\right] \tag{42}$$

Substitute (42) into (8).

$$\frac{d^2}{dt^2} T(t) + \left[\beta_n{}^4 \left(\frac{EI}{\rho}\right)\right] T(t) = 0 \tag{43}$$

Equation (43) is satisfied by

$$T(t) = b_1 \sin\left[\left(\beta_n{}^2 \sqrt{\frac{EI}{\rho}}\right) t\right] + b_2 \cos\left[\left(\beta_n{}^2 \sqrt{\frac{EI}{\rho}}\right) t\right] = 0 \tag{44}$$

The natural frequency term ω_n is thus

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \tag{45}$$

Substitute the value for the fundamental frequency from Table 1.

$$\omega_1 = \left[\frac{1.87510}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{46}$$

$$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{47}$$

Find the eigenvectors.

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{48}$$

$$a_4 = -a_2 \tag{49}$$

$$a_3 = -a_1 \tag{50}$$

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) - a_1 \sin(\beta x) - a_2 \cos(\beta x) \tag{51}$$

$$Y(x) = a_1 [\sinh(\beta x) - \sin(\beta x)] + a_2 [\cosh(\beta x) - \cos(\beta x)] \tag{52}$$

$$a_1[\sinh(\beta L)+\sin(\beta L)]+a_2[\cosh(\beta L)+\cos(\beta L)]=0 \tag{53}$$

$$a_2 = -a_1 \frac{[\sinh(\beta L)+\sin(\beta L)]}{[\cosh(\beta L)+\cos(\beta L)]} \tag{54}$$

$$a_1 - a_2 \frac{[\cosh(\beta L)+\cos(\beta L)]}{[\sinh(\beta L)+\sin(\beta L)]} \tag{55}$$

$$Y(x) = a_2\left\{ [\cosh(\beta x)-\cos(\beta x)] - \frac{[\cosh(\beta L)+\cos(\beta L)]}{[\sinh(\beta L)+\sin(\beta L)]}[\sinh(\beta x)-\sin(\beta x)]\right\} \tag{56}$$

Eigenvectors with arbitrary scale.

$$Y(x) =$$

$$\hat{a}_n\left\{ [\cosh(\beta_n x)-\cos(\beta_n x)] - \frac{[\cosh(\beta_n L)+\cos(\beta_n L)]}{[\sinh(\beta_n L)+\sin(\beta_n L)]}[\sinh(\beta_n x)-\sin(\beta_n x)]\right\} \tag{57}$$

Again, the β_n terms are given in Table 1.

The eigenvectors are normalized such that

$$\int_0^L \rho Y_n{}^2(x)\,dx = 1 \tag{58}$$

Normalize with respect to mass. The leading coefficient is 1 for each mode. Thus the eigenvectors normalized with respect to mass are

$$Y_1(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{[\cosh(\beta_1 x)-\cos(\beta_1 x)] - 0.73410\,[\sinh(\beta_1 x)-\sin(\beta_1 x)]\right\} \tag{59}$$

$$Y_2(x) = \left\{\frac{1}{\sqrt{\rho L}}\right\}\left\{[\cosh(\beta_2 x)-\cos(\beta_2 x)] - 1.01847\,[\sinh(\beta_2 x)-\sin(\beta_2 x)]\right\} \tag{60}$$

$$Y_3(x) = \left\{ \frac{1}{\sqrt{\rho L}} \right\} \left\{ \left[\cosh(\beta_3 x) - \cos(\beta_3 x) \right] - 0.99922 \left[\sinh(\beta_3 x) - \sin(\beta_3 x) \right] \right\} \quad (61)$$

$$Y_4(x) = \left\{ \frac{1}{\sqrt{\rho L}} \right\} \left\{ \left[\cosh(\beta_4 x) - \cos(\beta_4 x) \right] - 1.00003 \left[\sinh(\beta_4 x) - \sin(\beta_4 x) \right] \right\} \quad (62)$$

ρ and L are numerical values only. Y is non-dimensional. The units must be consistent, however.

Again, the β_n values for equations (59) through (62) are given in Table 1.

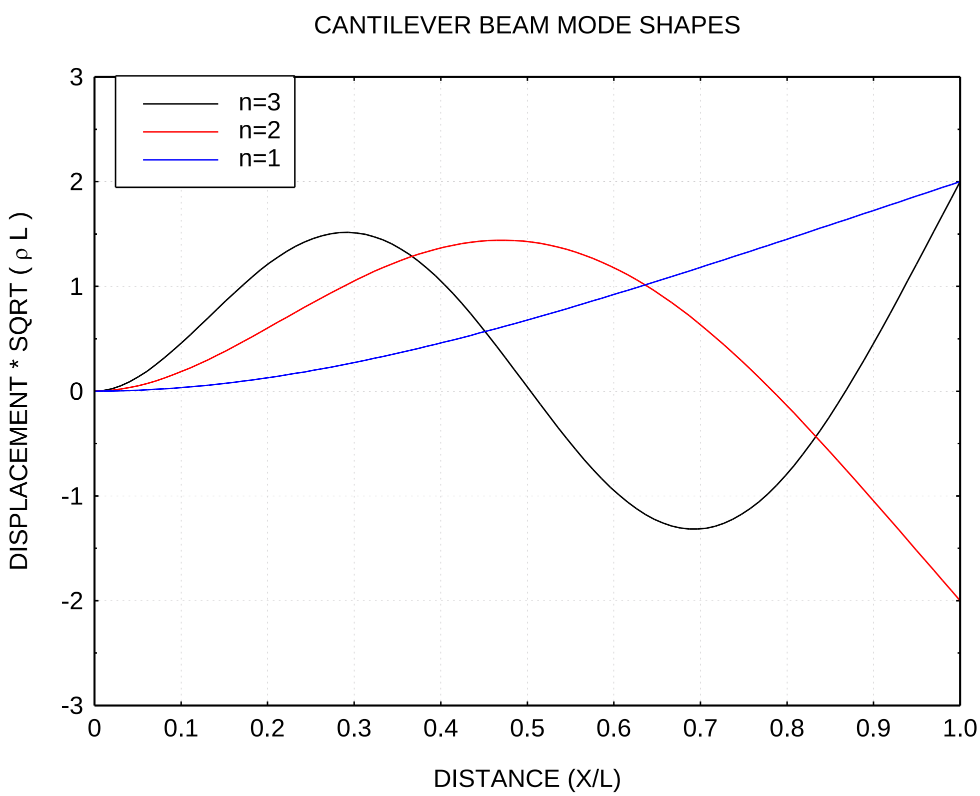

Figure 2.

The first three mode shapes are plotted in Figure 2.

Participation Factors

The participation factors for constant mass density are

$$\Gamma_n = \rho \int_0^L Y_n(x)\, dx \tag{63}$$

The participation factors from a numerical calculation are

$$\Gamma_1 = 0.7830 \sqrt{\rho L} \tag{64}$$

$$\Gamma_2 = 0.4339 \sqrt{\rho L} \tag{65}$$

$$\Gamma_3 = 0.2544 \sqrt{\rho L} \tag{66}$$

$$\Gamma_4 = 0.1818 \sqrt{\rho L} \tag{67}$$

The participation factors are non-dimensional.

Effective Modal Mass

The effective modal mass is

$$m_{eff,n} = \frac{\left[\int_0^L m(x)\, Y_n(x) dx \right]^2}{\int_0^L m(x) [Y_n(x)]^2\, dx} \tag{68}$$

The eigenvectors are already normalized such that

$$\int_0^L m(x) [Y_n(x)]^2\, dx = 1 \tag{69}$$

Thus,

$$m_{eff,n} = [\Gamma_n]^2 = \left[\int_0^L m(x)\, Y_n(x) dx \right]^2 \tag{70}$$

The effective modal mass values are obtained numerically.

$$m_{eff,1} = 0.6131\, \rho L \tag{71}$$

$$m_{eff,2} = 0.1883\ \rho L \tag{72}$$

$$m_{eff,3} = 0.06474\ \rho L \tag{73}$$

$$m_{eff,4} = 0.03306\ \rho L \tag{74}$$

The effective modal mass terms have units of mass.

Response to Base Excitation

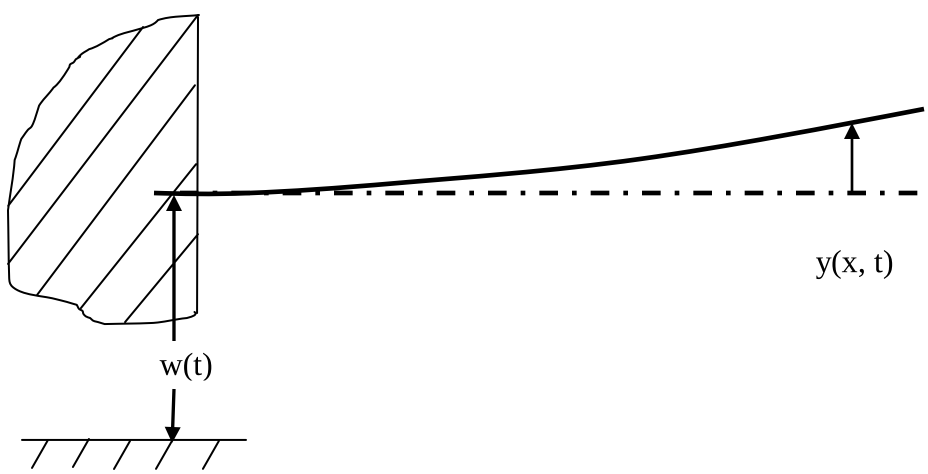

Figure 3.

The forced response equation for a beam with base motion is taken from Reference 1, page 345.

y(x,t) is the relative displacement.

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = -\rho\frac{\partial^2 w}{\partial t^2} \tag{75}$$

where w is the base displacement.

The term on the right-hand-side is the inertial force per unit length.

$$y(x,t) = \sum_{n=1}^{m} Y_n(x) T_n(t) \tag{76}$$

$$EI \frac{\partial^4}{\partial x^4}\left[\sum_{n=1}^{m} Y_n(x) T_n(t)\right] + \rho \frac{\partial^2}{\partial t^2}\left[\sum_{n=1}^{m} Y_n(x) T_n(t)\right] = -\rho \frac{\partial^2 w}{\partial t^2} \tag{77}$$

$$EI\left[\sum_{n=1}^{m} T_n(t) \frac{\partial^4}{\partial x^4} Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x) \frac{\partial^2}{\partial t^2} T_n(t)\right] = -\rho \frac{\partial^2 w}{\partial t^2} \tag{78}$$

$$EI\left[\sum_{n=1}^{m} T_n(t) \frac{d^4}{dx^4} Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x) \frac{d^2}{dt^2} T_n(t)\right] = -\rho \frac{d^2 w}{dt^2} \tag{79}$$

$$\frac{d^4}{dx^4} Y_n(x) = \beta_n{}^4 Y_n(x) \tag{80}$$

$$EI\left[\sum_{n=1}^{m} \beta_n{}^4 T_n(t) Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x) \frac{d^2}{dt^2} T_n(t)\right] = -\rho \frac{d^2 w}{dt^2} \tag{81}$$

$$EI\,\beta_n{}^4\left[\sum_{n=1}^{m} T_n(t) Y_n(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x) \frac{d^2}{dt^2} T_n(t)\right] = -\rho \frac{d^2 w}{dt^2} \tag{82}$$

Multiply each term by $Y_p(x)$.

$$EI\,\beta_n{}^4\left[\sum_{n=1}^{m} T_n(t) Y_n(x) Y_p(x)\right] + \rho\left[\sum_{n=1}^{m} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t)\right] = -\rho \frac{d^2 w}{dt^2} Y_p(x) \tag{83}$$

Integrate with respect to length.

$$\int_0^L \left\{ EI\beta_n{}^4 \left[\sum_{n=1}^{m} T_n(t) Y_n(x) Y_p(x) \right] + \rho \left[\sum_{n=1}^{m} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t) \right] \right\} dx$$

$$= -\int_0^L \rho \frac{d^2 w}{dt^2} Y_p(x) dx$$

(84)

$$EI\beta_n{}^4 \int_0^L \left[\sum_{n=1}^{m} T_n(t) Y_n(x) Y_p(x) \right] dx + \rho \int_0^L \left[\sum_{n=1}^{m} Y_n(x) Y_p(x) \frac{d^2}{dt^2} T_n(t) \right] dx$$

$$= -\int_0^L \rho \frac{d^2 w}{dt^2} Y_p(x) dx$$

(85)

$$EI\beta_n{}^4 \sum_{n=1}^{m} \left\{ T_n(t) \int_0^L Y_n(x) Y_p(x)\, dx \right\} + \rho \sum_{n=1}^{m} \left\{ \frac{d^2}{dt^2} T_n(t) \int_0^L Y_n(x) Y_p(x) dx \right\}$$

$$= -\rho \frac{d^2 w}{dt^2} \int_0^L Y_p(x) dx$$

(86)

$$\frac{EI\beta_n{}^4}{\rho} \sum_{n=1}^{m} \left\{ T_n(t) \int_0^L \rho Y_n(x) Y_p(x)\, dx \right\} + \sum_{n=1}^{m} \left\{ \frac{d^2}{dt^2} T_n(t) \int_0^L \rho Y_n(x) Y_p(x) dx \right\}$$

$$= -\frac{d^2 w}{dt^2} \int_0^L \rho Y_p(x) dx$$

(87)

The eigenvectors are orthogonal such that

$$\int_0^L \rho Y_n(x) Y_p(x)\, dx = 0 \qquad \text{for } n \neq p \tag{88}$$

$$\int_0^L \rho Y_n(x) Y_p(x)\, dx = 1 \qquad \text{for } n = p \tag{89}$$

$$\frac{d^2}{dt^2} T_n(t) + \frac{EI\,\beta_n{}^4}{\rho} T_n(t) = -\frac{d^2 w}{dt^2} \int_0^L \rho Y_p(x) dx \tag{90}$$

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{\rho}} \tag{91}$$

$$\omega_n{}^2 = \frac{EI}{\rho} \beta_n{}^4 \tag{92}$$

$$\frac{d^2}{dt^2} T_n(t) + \omega_n{}^2\, T_n(t) = -\frac{d^2 w}{dt^2} \int_0^L \rho Y_p(x) dx \tag{93}$$

$$\frac{d^2}{dt^2} T_n(t) + \omega_n{}^2\, T_n(t) = -\Gamma_n \frac{d^2 w}{dt^2} \tag{94}$$

Add a damping term.

$$\frac{d^2}{dt^2} T_n(t) + 2\xi_n \omega_n \dot{T}_n(t) + \omega_n{}^2\, T_n(t) = -\Gamma_n \frac{d^2 w}{dt^2} \tag{95}$$

$$\ddot{T}_n(t) + 2\xi_n \omega_n \dot{T}(t) + \omega_n{}^2\, T_n(t) = -\Gamma_n \ddot{w}(t) \tag{96}$$

Steady-State Solution

Take a Fourier transform of both sides of (96).

$$\int_{-\infty}^{\infty}\left\{\ddot{T}_n(t) + 2\xi_n\omega_n\dot{T}(t) + \omega_n{}^2\ T_n(t)\right\}\exp(-j\omega t)dt = -\ \Gamma_n\int_{-\infty}^{\infty}\ddot{w}(t)\exp(-j\omega t)dt \tag{97}$$

$$\int_{-\infty}^{\infty}\ddot{T}_n(t)\exp(-j\omega t)dt + 2\xi_n\omega_n\int_{-\infty}^{\infty}\dot{T}(t)\exp(-j\omega t)dt + \omega_n{}^2\int_{-\infty}^{\infty}T_n(t)\exp(-j\omega t)dt$$

$$= -\ \Gamma_n\int_{-\infty}^{\infty}\ddot{w}(t)\exp(-j\omega t)dt$$

(98)

$$\int_{-\infty}^{\infty}\dot{T}_n(t)\ \exp(-j\omega t)dt = j\omega\int_{-\infty}^{\infty}T_n(t)\ \exp(-j\omega t)dt \tag{99}$$

$$\int_{-\infty}^{\infty}\ddot{T}_n(t)\ \exp(-j\omega t)dt = -\omega^2\int_{-\infty}^{\infty}T_n(t)\ \exp(-j\omega t)dt \tag{100}$$

$$-\omega^2\int_{-\infty}^{\infty}T_n(t)\ \exp(-j\omega t)dt + j2\xi_n\omega_n\omega\int_{-\infty}^{\infty}T_n(t)\ \exp(-j\omega t)dt + \omega_n{}^2\int_{-\infty}^{\infty}T_n(t)\exp(-j\omega t)dt$$

$$= -\ \Gamma_n\int_{-\infty}^{\infty}\ddot{w}(t)\exp(-j\omega t)dt$$

(101)

$$\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]\int_{-\infty}^{\infty}T_n(t)\ \exp(-j\omega t)dt = -\ \Gamma_n\int_{-\infty}^{\infty}\ddot{w}(t)\exp(-j\omega t)dt \tag{102}$$

$$Z_n(\omega) = \int_{-\infty}^{\infty}T_n(t)\exp(-j\omega t)dt \tag{103}$$

$$\dot{Z}_n(\omega) = \int_{-\infty}^{\infty} \dot{T}_n(t) \exp(-j\omega t) dt$$

$$= j\omega \int_{-\infty}^{\infty} T_n(t) \exp(-j\omega t) dt$$

$$= j\omega Z_n(\omega) \tag{104}$$

$$\ddot{Z}_n(\omega) = \int_{-\infty}^{\infty} \ddot{T}_n(t) \exp(-j\omega t) dt$$

$$= -\omega^2 \int_{-\infty}^{\infty} T_n(t) \exp(-j\omega t) dt$$

$$= -\omega^2 Z_n(\omega) \tag{105}$$

$$\ddot{W}(\omega) = \int_{-\infty}^{\infty} \ddot{w}(t) \exp(-j\omega t) dt \tag{106}$$

$$\ddot{Z}_n(\omega) + j2\xi_n\omega_n \dot{Z}_n(\omega) + \omega_n{}^2 Z_n(\omega) = -\Gamma_n \ddot{W}(\omega) \tag{107}$$

Substitute equation (103) into (102).

$$\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right] Z_n(\omega) = -\Gamma_n \ddot{W}(\omega) \tag{108}$$

$$Z_n(\omega) = \left[\frac{-1}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]}\right] \Gamma_n \ddot{W}(\omega) \tag{109}$$

$$\dot{Z}_n(\omega) = \left[\frac{-j\omega}{\left[\left(\omega_n{}^2 - \omega^2\right) + j2\xi_n\omega_n\omega\right]}\right] \Gamma_n \ddot{W}(\omega) \tag{110}$$

$$Y(x,\omega) = \sum_{n=1}^{m} Y_n(x) Z_n(\omega) \tag{111}$$

The Fourier transform of the relative displacement is

$$Y(x,\omega) = \ddot{W}(\omega) \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{(\omega_n^2 - \omega^2) + j2\xi_n \omega \omega_n} \right\} \tag{112}$$

The frequency response function relating the relative displacement to the base acceleration is

$$H_{rd}(x,\omega) = \frac{Y(x,\omega)}{\ddot{W}(\omega)} \tag{113}$$

$$H_{rd}(x,\omega) = \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{(\omega_n^2 - \omega^2) + j2\xi_n \omega \omega_n} \right\} \tag{114}$$

The frequency response function relating the relative velocity to the base acceleration is

$$H_{rv}(x,\omega) = j\omega H_{rd}(x,\omega) \tag{115}$$

$$H_{rv}(x,\omega) = j\omega \sum_{n=1}^{m} \left\{ \frac{-\Gamma_n Y_n(x)}{(\omega_n^2 - \omega^2) + j2\xi_n \omega \omega_n} \right\} \tag{116}$$

The frequency response function relating the relative acceleration to the base acceleration is

$$H_{ra}(x,\omega) = \omega^2 \sum_{n=1}^{m} \left\{ \frac{\Gamma_n Y_n(x)}{(\omega_n^2 - \omega^2) + j2\xi_n \omega\omega_n} \right\} \tag{117}$$

The absolute acceleration $A(x,\omega)$ is

$$A(x,\omega) = H_{ra}(x,\omega)\ddot{W}(\omega) + \ddot{W}(\omega) \tag{118}$$

$$A(x,\omega) = [H_{ra}(x,\omega) + 1]\ddot{W}(\omega) \tag{119}$$

The frequency response function relating the absolute acceleration to the base acceleration is

$$H_{aa}(x,\omega) = H_{ra}(x,\omega) + 1 \tag{120}$$

Example

Consider a beam with the following properties:

Cross-Section	Circular
Boundary Conditions	Fixed-Free
Material	Aluminum

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	24 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.471 lbm (1.22E-03 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05 for all modes

The analysis is performed using Matlab script: cantilever_beam.m. The normal modes results are given in Table 2. Again, both the mode shape and participation factor are considered as dimensionless, but they must be consistent with respect to one another.

Table 2. Natural Frequency Results, Fixed-Free Beam

Mode	fn (Hz)	Participation Factor	Effective Modal Mass (lbf sec^2/in)	Effective Modal Mass (lbm)
1	23.86	0.02736	0.00074837	0.289
2	149.53	0.01516	0.00022982	0.089
3	418.69	0.00889	7.9028e-05	0.031
4	820.47	0.00635	4.0361e-05	0.016

The frequency response function plots for relative displacement and absolute acceleration are given in Figures 4 and 5, respectively.

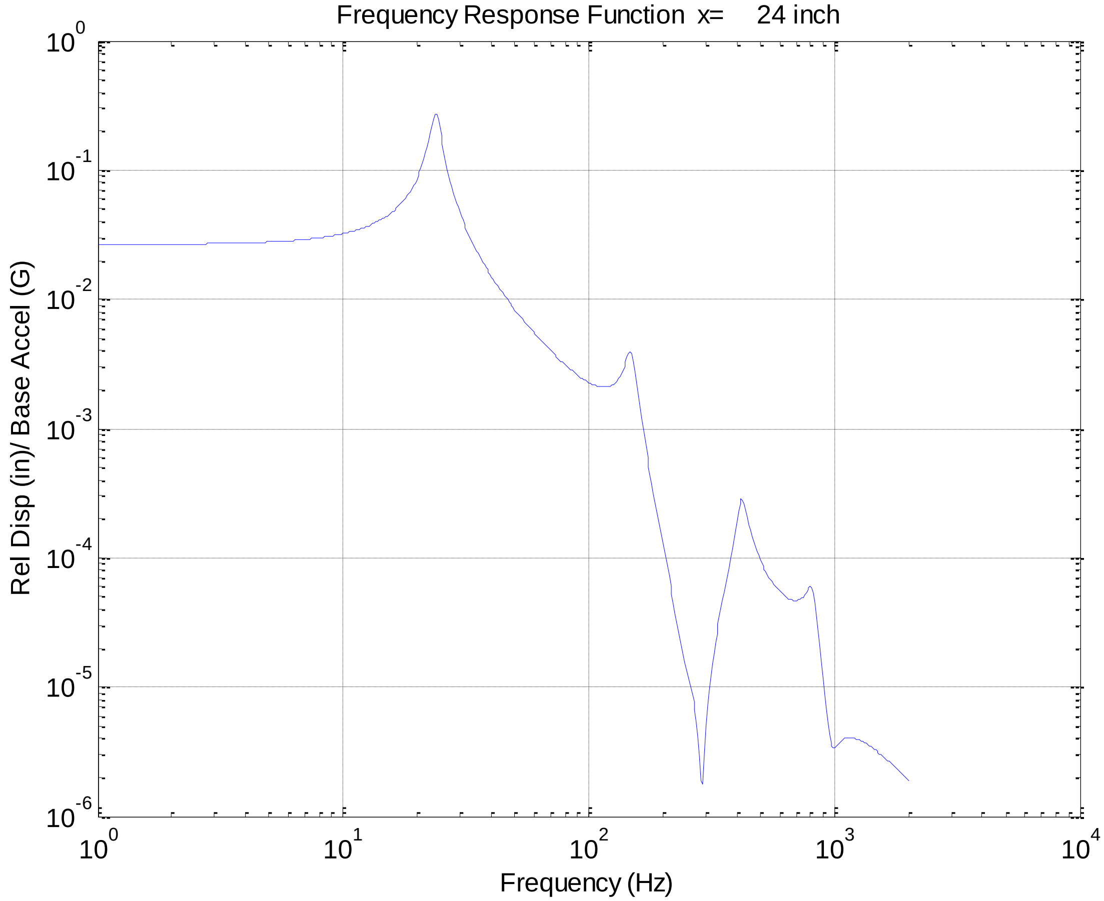
Frequency Response Function x= 24 inch
Rel Disp (in)/ Base Accel (G)
Frequency (Hz)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^0
10^1
10^2
10^3
10^4

Figure 4.

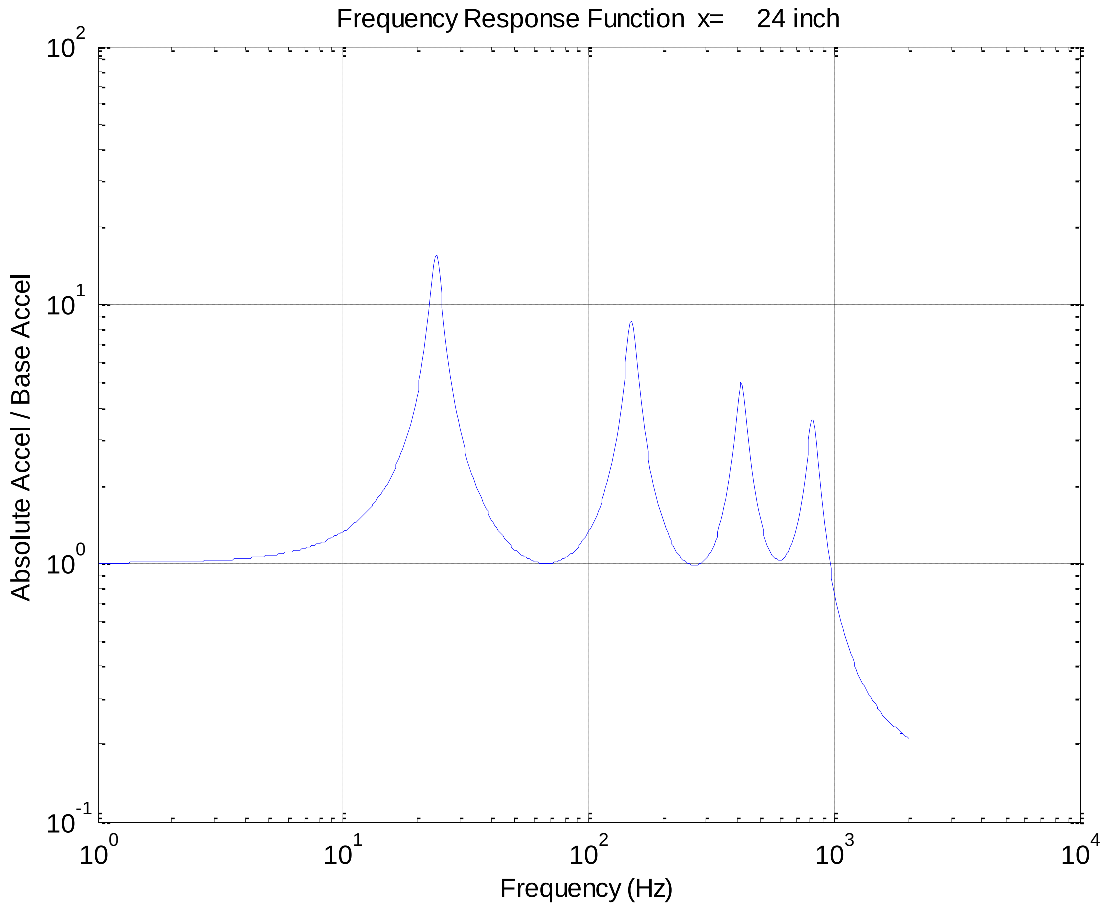
Frequency Response Function x= 24 inch
Absolute Accel / Base Accel
Frequency (Hz)
10^{2}
10^{1}
10^{0}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}

Figure 5.

Simple Response Calculation

Consider a 1 G sinusoidal input at 24 Hz.

The relative displacement magnitude at 24 Hz is 0.27 inch/G.

The relative displacement response at the free end of the beam is 0.27 inch.

The absolute acceleration magnitude at 24 Hz is 15.6 G/G.

The absolute acceleration response at the free end of the beam is 15.6 G.

Reference

1. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.